Correlation features of interacting bosons in one dimension

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Summary

Free Lieb-Liniger

arXiv:2104.10491

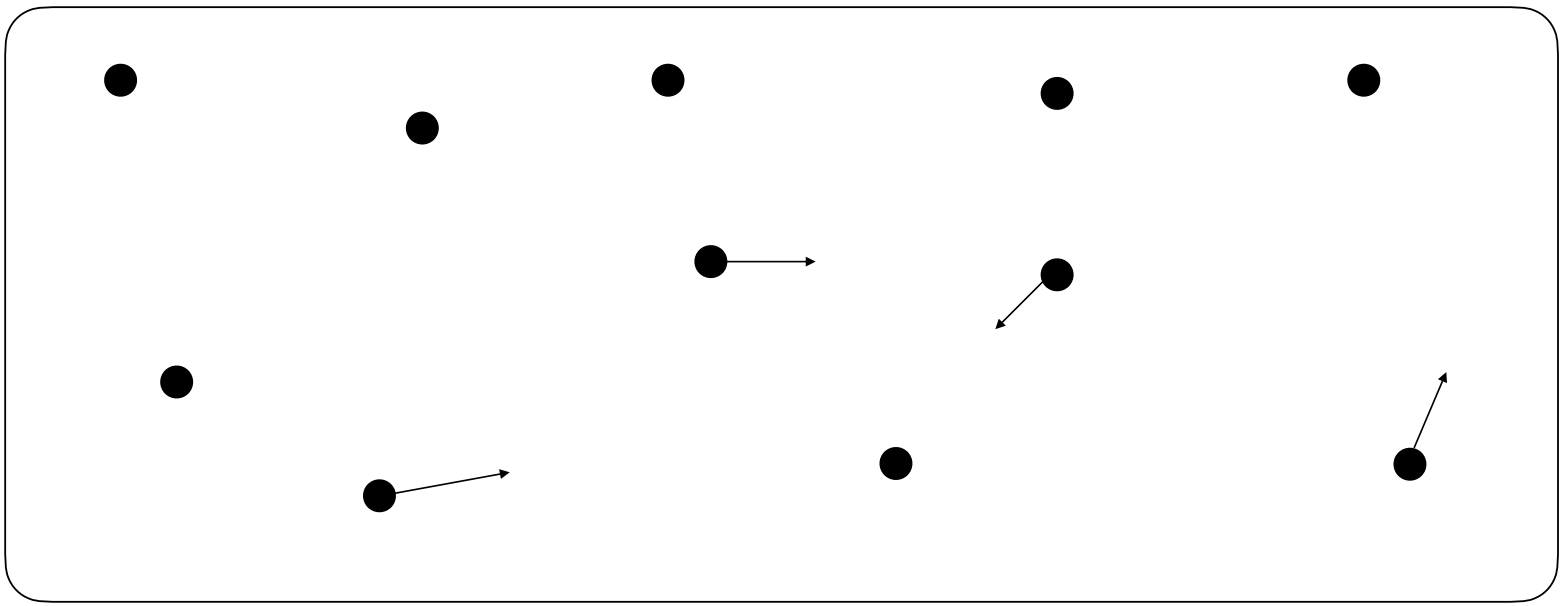
- One-body correlation function
- Form factors

Trapped Lieb-Liniger

arXiv:1908.08714

- Asymptotic behavior of the momentum distribution
- The Tonks-Girardeau limit
- Scaling properties
  - Zero-temperature scaling
  - Large-temperature scaling
  - Generalized scaling conjecture

Wonders of one-dimensional world





Some physical peculiarities

- Landau-Fermi liquid theory, describing interacting electrons, breaks down in one dimension[1]: 1d analogue is the Tomonaga-Luttinger theory [2];
- There is no $U(1)$-symmetry breaking in 1d (HMW theorem);
- No BEC occurs for infinite systems at $T > 0$ in $d \leq 2$ (spatial confinement restores BEC occurrence);
- ...

[1] LANDAU, L. D. The theory of a Fermi liquid. Journal of Experimental and Theoretical Physics, v. 30, n. 6, p. 920, 1956

[2] LUTTINGER, J. M. An exactly soluble model of a many-fermion system. Journal of Mathematical Physics, v. 4, n. 9, p. 1154-1162, 1963

Section 1

Free Lieb-Liniger

arXiv:2104.10491

Interacting bosonic particles in 1d

- Point-like interaction:

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2c\sum_{i<j}\delta(x_i - x_j). \tag{1}$$

- Solved by Lieb and Liniger in 1960 via Bethe ansatz considerations (plane waves with some permutation terms...)
- Upon imposing periodic boundary conditions, the rapidities are provided by the Bethe equations

$$\lambda_j + \frac{2}{L}\sum_{\ell=1}^{N}\arctan\left(\frac{\lambda_j - \lambda_\ell}{c}\right) = \frac{2\pi}{L}I_j, \quad j = 1, \ldots, N, \tag{2}$$

where we identitfy I_j's as quantum numbers, that can be either integers (odd N) or half-odd integers (even N). The ground-state is given by the Fermi sea,

$$I_j = j - (N+1)/2, \quad j = 1, \ldots, N. \tag{3}$$

- The energy and momentum of a given eigenstate are $E_{\boldsymbol{\lambda}} = \sum_{j=1}^{N}\lambda_j^2, \qquad P_{\boldsymbol{\lambda}} = \sum_{j=1}^{N}\lambda_j.$

One-body function

- One-body correlation function over the ground state

$$g(x,t) = \langle \boldsymbol{\lambda} | \Psi^\dagger(x,t) \Psi(0,0) | \boldsymbol{\lambda} \rangle, \tag{4}$$

- $$g(x,t) = \sum_{\boldsymbol{\mu} \in \mathcal{H}_{N-1}} \langle \boldsymbol{\lambda} | \Psi^\dagger(x,t) | \boldsymbol{\mu} \rangle \langle \boldsymbol{\mu} | \Psi(0,0) | \boldsymbol{\lambda} \rangle, \tag{5}$$

- Time evolution

$$g(x,t) = \sum_{\boldsymbol{\mu} \in \mathcal{H}_{N-1}} e^{-i(E_{\boldsymbol{\lambda}} - E_{\boldsymbol{\mu}})t + i(P_{\boldsymbol{\mu}} - P_{\boldsymbol{\lambda}})x} |\langle \boldsymbol{\mu} | \Psi(0) | \boldsymbol{\lambda} \rangle|^2, \tag{6}$$

where the space- and time-independent term, $|\langle \boldsymbol{\mu} | \Psi(0) | \boldsymbol{\lambda} \rangle|^2$, is the so-called form factor.

Form factors

The field operator form factor for the Lieb-Liniger, can somehow be computed (Algebraic Bethe Ansatz? Quantum Inverse Scattering Method?), yielding

$$|\langle \boldsymbol{\mu}|\Psi(0)|\boldsymbol{\lambda}\rangle|^2 = c^{2N-1}\frac{\prod_{j>k=1}^{N}\left(\lambda_{jk}^2+c^2\right)^2}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\left(\lambda_j-\mu_k\right)^2}\frac{\det^2_{N-1}U(\boldsymbol{\mu},\boldsymbol{\lambda})}{\|\boldsymbol{\mu}\|^2\|\boldsymbol{\lambda}\|^2}, \tag{7}$$

where we abbreviate $\lambda_{jk}\equiv\lambda_j-\lambda_k$. The entries of the $(N-1)\times(N-1)$ matrix $U(\boldsymbol{\mu},\boldsymbol{\lambda})$ are given by

$$U_{jk}(\boldsymbol{\mu},\boldsymbol{\lambda}) = \delta_{jk}\frac{V_j^+-V_j^-}{i}+\frac{\prod_{a=1}^{N-1}(\mu_a-\lambda_j)}{\prod_{a\neq j}^{N}(\lambda_a-\lambda_j)}\left[K(\lambda_j-\lambda_k)-K(\lambda_N-\lambda_k)\right], \quad j,k=1,\ldots,N-1, \tag{8}$$

with

$$V_j^{\pm}=\frac{\prod_{a=1}^{N-1}(\mu_a-\lambda_j\pm ic)}{\prod_{a=1}^{N}(\lambda_a-\lambda_j\pm ic)}. \tag{9}$$

One-body function of TG gas

- Strongly repulsive regime $c \to \infty$: Tonks-Girardeau gas:
  1. Strong repulsion between particles plays the role of the Pauli exclusion principle;
  2. The bosonic system can be mapped into the fermionic one, *fermionization*;
  3. Energy spectra are equal, whereas wave functions are not, but there is a correspondence.
- A milestone is the exact result from Lenard [3]

$$g(x) = \det_N(I + V_1 + V_2) - \det_N(I + V_1), \tag{10}$$

where I is the identity matrix and

$$V_1^{ij} = -\frac{4}{L}\frac{\sin(x(k_i - k_j)/2)}{(k_i - k_j)}, \qquad V_2^{ij} = \frac{1}{L}e^{-ix(k_i+k_j)/2}, \tag{11}$$
$$i, j = 1, \ldots, N.$$

[3] A. Lenard, *Momentum Distribution in the Ground State of the One-Dimensional system of Impenetrable Bosons*, *J. Math. Phys.* **5** (1964) 930–943.

Form factors in TG gas

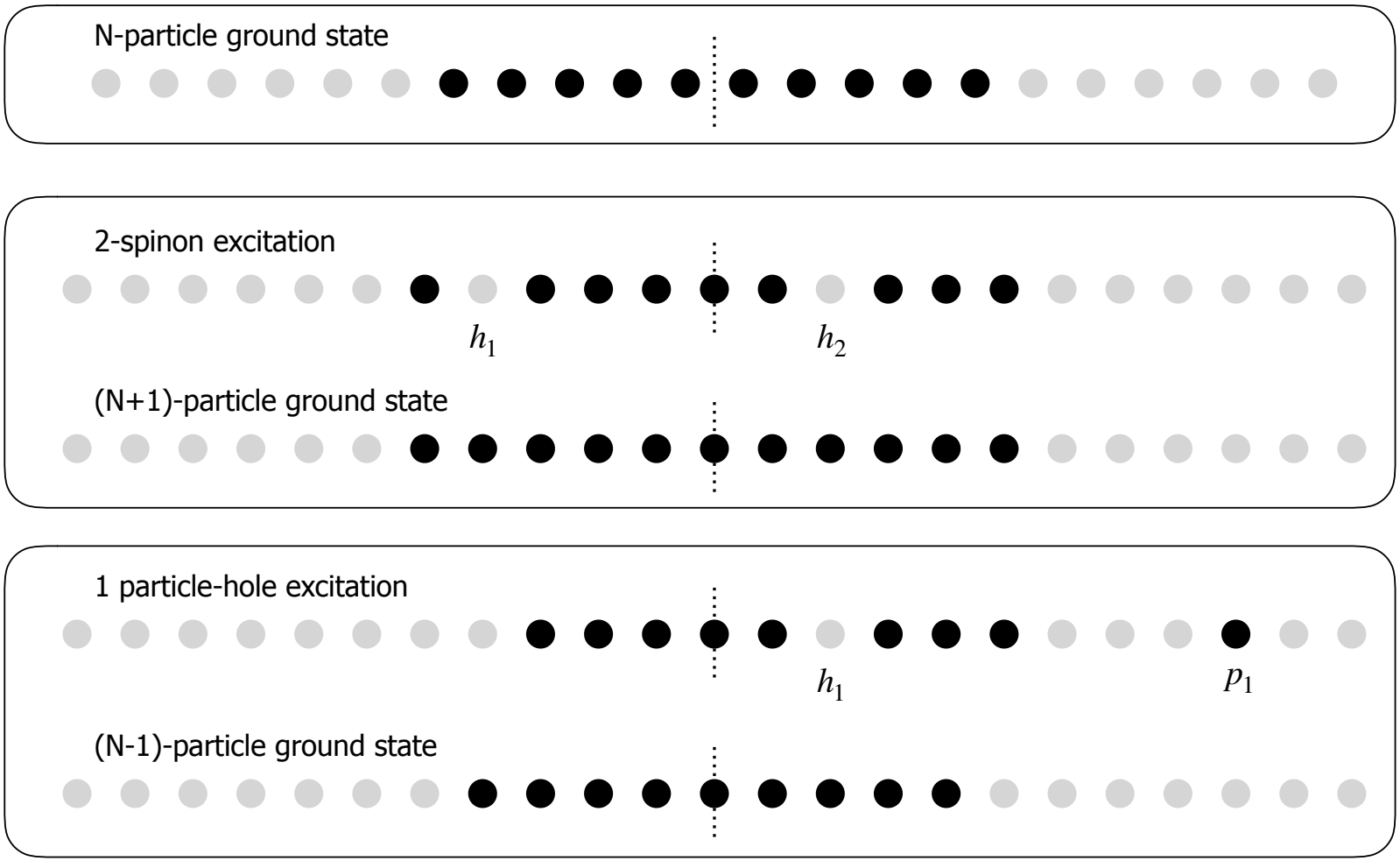
- At $c \to \infty$ the Bethe equations (2) decouple to simple quantization conditions

$$\lambda_j = \frac{2\pi}{L} I_j, \qquad j = 1, \ldots, N; \tag{12}$$

- The limit $c \to \infty$ of the form factor (7) is given by

$$|\langle \boldsymbol{\mu}|\Psi(0)|\boldsymbol{\lambda}\rangle|^2 = \frac{1}{2}\left(\frac{2}{L}\right)^{2N-1} \frac{\prod_{j>k=1}^{N}(\lambda_j-\lambda_k)^2 \prod_{j>k=1}^{N-1}(\mu_j-\mu_k)^2}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}(\lambda_j-\mu_k)^2}. \tag{13}$$

Elementary excitations



Momentum aftermath

- Momentum change of the system:

$$\Delta k_{2sp} \equiv k_{2sp}^{(N-1)} - k_{GS}^{(N+1)} = \frac{2\pi}{L}\left(\sum_{\substack{j=1 \\ j\neq h_1,h_2}}^{N+1} I_j^{(N+1)} - \sum_{j=1}^{N+1} I_j^{(N+1)}\right) \quad (14)$$
$$= -\frac{2\pi}{L}\left(I_{h_1}^{(N+1)} + I_{h_2}^{(N+1)}\right).$$

- We define the momentum of the two-spinon excitation as the difference between the excited and the ground states with the same number of particles,

$$k_{2sp} \equiv k_{2sp}^{(N-1)} - k_{GS}^{(N-1)} = \frac{2\pi}{L}\left(\sum_{\substack{j=1 \\ j\neq h_1,h_2}}^{N+1} I_j^{(N+1)} - \sum_{j=1}^{N-1} I_j^{(N-1)}\right). \quad (15)$$

Thence, we have that

$$k_{2sp} = -\frac{2\pi}{L}\left(I_{h_1}^{(N+1)} + I_{h_2}^{(N+1)}\right). \quad (16)$$

- The momentum of the excitation is equal to the momentum change.

Energy aftermath

- Energy change in the system:

$$\Delta\omega_{2sp} \equiv \omega_{2sp}^{(N-1)} - \omega_{GS}^{(N+1)} = \left(\frac{2\pi}{L}\right)^2 \left(\sum_{\substack{j=1 \\ j\neq h_1,h_2}}^{N+1} \left(I_j^{(N+1)}\right)^2 - \sum_{j=1}^{N+1} \left(I_j^{(N+1)}\right)^2 \right)$$
$$= -\left(\frac{2\pi}{L}\right)^2 \left[\left(I_{h_1}^{(N+1)}\right)^2 + \left(I_{h_2}^{(N+1)}\right)^2 \right]. \tag{17}$$

- Excitation energy:

$$\omega_{2sp} \equiv \omega_{2sp}^{(N-1)} - \omega_{GS}^{(N-1)} = \left(\frac{2\pi}{L}\right)^2 \left(\sum_{\substack{j=1 \\ j\neq h_1,h_2}}^{N+1} \left(I_j^{(N+1)}\right)^2 - \sum_{j=1}^{N-1} \left(I_j^{(N-1)}\right)^2 \right)$$
$$= -\left(\frac{2\pi}{L}\right)^2 \left[\left(I_{h_1}^{(N+1)}\right)^2 + \left(I_{h_2}^{(N+1)}\right)^2 - \frac{N^2}{2} \right]. \tag{18}$$

- The energy of the excitation is greater than the energy change

$$\omega_{2sp} = \Delta\omega_{2sp} + 2\epsilon_F. \tag{19}$$

Form factors of two spinons

- A generic excited state consists of two spinons parametrized by the positions of the two holes, (h_1, h_2), and m particle-hole excitations, each described by a pair (p_j, h_j) with $j = 3, \ldots, m+2$. In this fashion the form factor (13) becomes

$$|\langle \boldsymbol{\mu}|\Psi(0)|\boldsymbol{\lambda}\rangle|^2 = \Omega(L,N) \times \prod_{a=1}^{m+2} \frac{\prod_{j=1}^{N}(\lambda_j - h_a)^2}{\prod_{j=1}^{N+1}(\bar{\mu}_j - h_a)^2} \prod_{a=3}^{m+2} \frac{\prod_{j=1}^{N+1}(\bar{\mu}_j - p_a)^2}{\prod_{j=1}^{N}(\lambda_j - p_a)^2} \times \frac{\prod_{a>b=3}^{m+2}(h_a - h_b)^2 \prod_{a>b=1}^{m+2}(p_a - p_b)^2}{\prod_{a=1}^{m+2}\prod_{b=3}^{m+2}(h_a - p_b)^2}. \tag{20}$$

where

$$\Omega(N,L) = \frac{1}{2}\left(\frac{2}{L}\right)^{2N-1} \frac{\prod_{j>k=1}^{N}(\lambda_j - \lambda_k)^2 \prod_{j>k=1}^{N+1}(\bar{\mu}_j - \bar{\mu}_k)^2}{\prod_{j=1}^{N}\prod_{k=1}^{N+1}(\lambda_j - \bar{\mu}_k)^2} = \frac{L}{4} G^4(3/2) \left[\frac{G(N+1)G(N+2)}{G^2(N+3/2)}\right]^2, \tag{21}$$

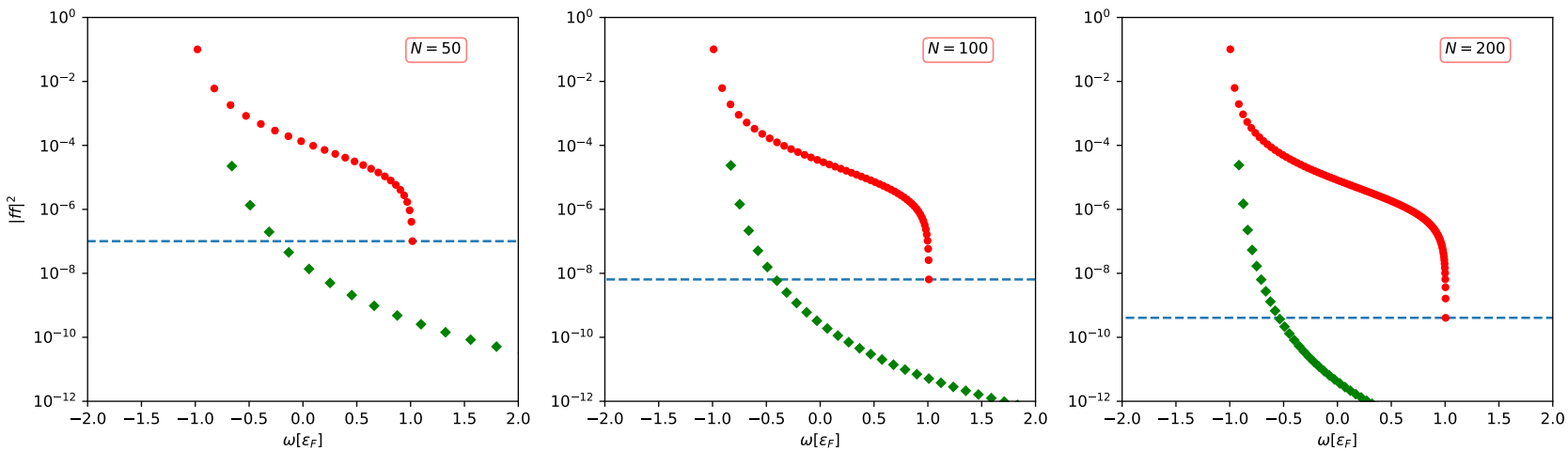
is a factor which is independent of the excited state.

Further particle-hole excitations

- Consider a further particle-hole excitation over the excited state $|\boldsymbol{\mu}\rangle$ (2sp+$m$ph), i.e., $|\boldsymbol{\mu} + (p_{m+3}, h_{m+3})\rangle$. It is possible to show that ratio between the respective form factors is less than 1,

$$|\langle \boldsymbol{\mu} + (p_{m+3}, h_{m+3})|\Psi(0)|\boldsymbol{\lambda}\rangle|^2 < |\langle \boldsymbol{\mu}|\Psi(0)|\boldsymbol{\lambda}\rangle|^2. \qquad (22)$$

- Form factors for two classes of zero-momentum excitations. The form factors with two-spinon excitations (red) are larger than the form factors for 2ph on top of 2sp (green).



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Two-spinon one-body function

- The two-spinon one-body function is given by,

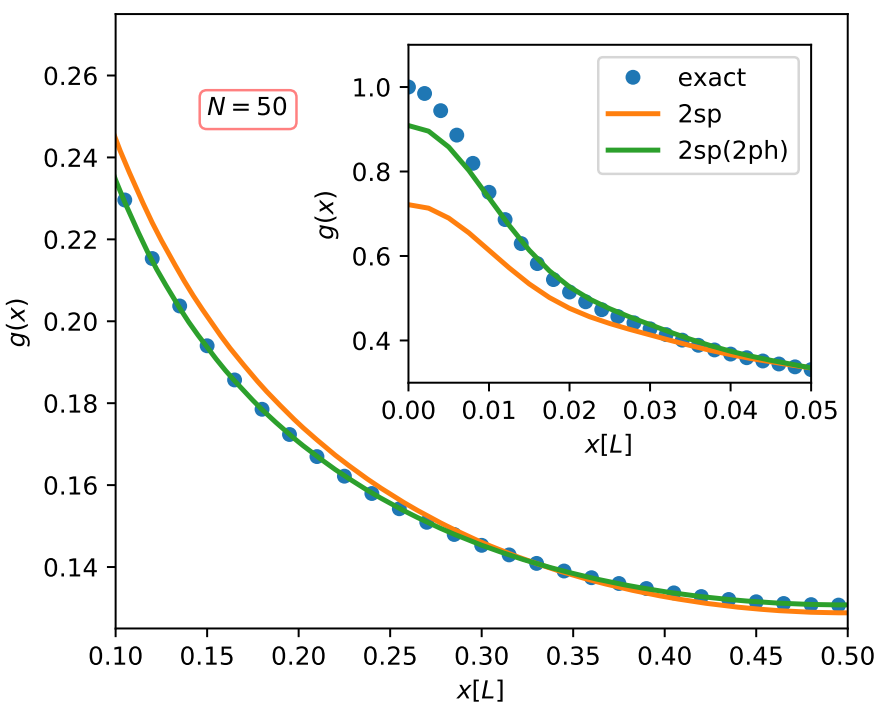
$$g_{2\mathrm{sp}}(x,t) = \sum_{|\alpha\rangle \in \mathcal{H}_{2sp}} e^{i\omega_{2\mathrm{sp}}(\alpha)t - ik_{2\mathrm{sp}}(\alpha)x} |\langle \alpha|\Psi(0)|\mathrm{GS}\rangle|^2. \tag{23}$$

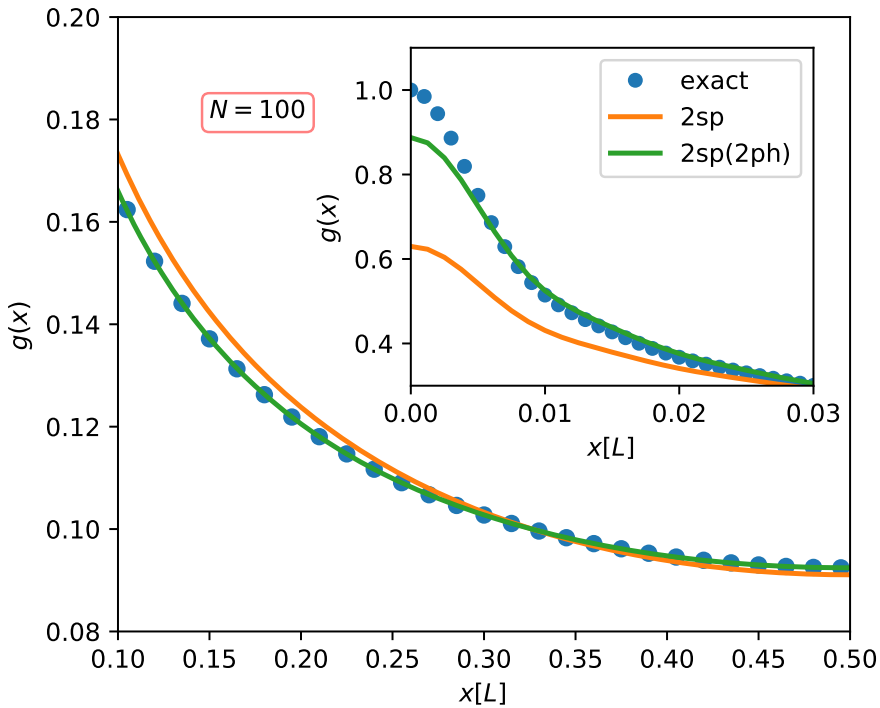
where the 2sp form factor is equivalent to (20) for $m = 0$, i.e.,

$$|\langle \boldsymbol{\mu}|\Psi(0)|\boldsymbol{\lambda}\rangle|^2 = \Omega(L,N) \times \frac{\prod_{j=1}^{N}(\lambda_j - h_1)^2}{\prod_{j=1}^{N+1}(\bar{\mu}_j - h_1)^2} \frac{\prod_{j=1}^{N}(\lambda_j - h_2)^2}{\prod_{j=1}^{N+1}(\bar{\mu}_j - h_2)^2} (h_1 - h_2)^2, \tag{24}$$

and the sum extends over possible choices of the two holes (h_1, h_2) in the $(N+1)$-particle ground state. The rapidities λ_j and $\bar{\mu}_j$ are respectively ground-state rapidities of systems with N and $N+1$ particles.

2sp one-body function





arXiv:2104.10491

Conclusive remarks

- We showed that the two-spinon excitation contributes with a large part to calculate the one-body function via the summation of form factor for the Tonks-Girardeau gas;
- At finite interaction, we performed numerical calculations (ABACUS) to evaluate the two-spinon contribution.

Section 2

Trapped Lieb-Liniger

arXiv:1908.08714

The interaction manifestation

- Trapped Lieb-Liniger gas of N particles:

$$\hat{H} = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + V(x_i) \right) + g \sum_{i<j} \delta(x_i - x_j). \tag{25}$$

- For any pair of particles (i, j):

$$\int_{-\varepsilon}^{+\varepsilon} \left[\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_{ij}^2} + V(x_{ij}) + g\delta(x_{ij}) - E \right) \Psi(x_1, x_2, \ldots, x_N) \right] dx_{ij} = 0$$

$$\overset{\varepsilon \to 0}{\Rightarrow} -\frac{\hbar^2}{2m} \frac{\partial \Psi}{\partial x_{ij}} \Bigg|_{-\varepsilon}^{+\varepsilon} + g\Psi(x_{ij} = 0) = 0. \tag{26}$$

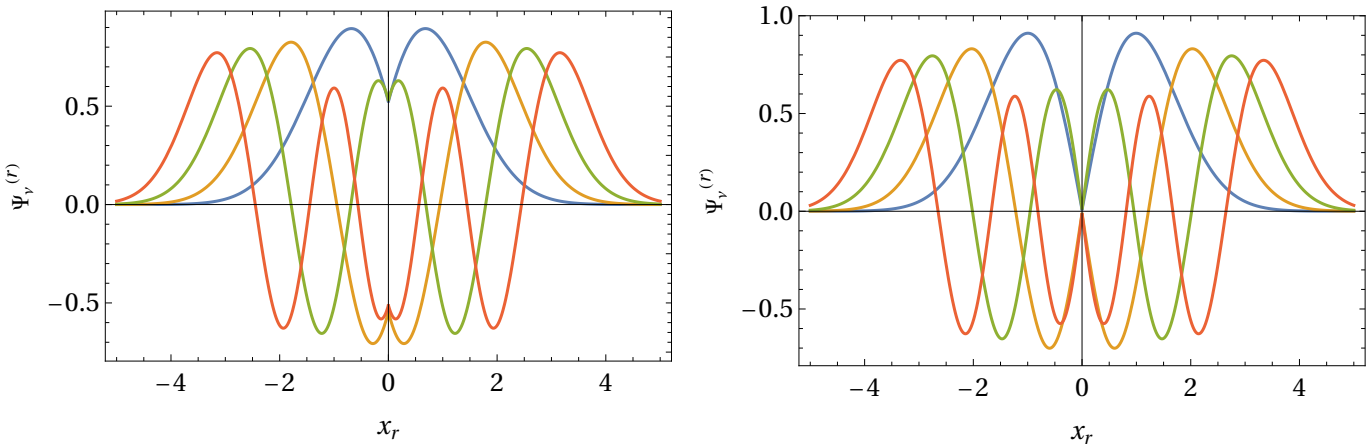
As $\varepsilon \to 0$, the contact interaction generates a condition given by

$$\left(\frac{\partial \Psi}{\partial x_i} - \frac{\partial \Psi}{\partial x_j} \right) \Bigg|_{x_i - x_j \to 0^+} - \left(\frac{\partial \Psi}{\partial x_i} - \frac{\partial \Psi}{\partial x_j} \right) \Bigg|_{x_i - x_j \to 0^-} = \frac{2mg}{\hbar^2} \Psi(x_i = x_j). \tag{27}$$

The two-body case

- For $N = 2$ and $V(x_i) = m\omega^2 x_i^2/2$ in (25), the solutions are

$$\psi_\nu^{(r)}(x_r) = \sqrt{\frac{1}{\mathcal{N}(\nu)}}\, \mathrm{e}^{-(x_r/a_0)^2/2}\, U\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{x_r^2}{a_0^2}\right). \tag{28}$$



(a) $\tilde{g} = 1$ (b) $\tilde{g} = 100$

- Its expansion around $x = 0$ is

$$\psi_\nu^{(r)}(x) \sim \sqrt{\frac{\pi}{\mathcal{N}(\nu)}} \left[\Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right)^{-1} - 2\,\Gamma\left(-\frac{\nu}{2}\right)^{-1} \frac{|x|}{a_0} + \mathcal{O}(x^2)\right]. \tag{29}$$

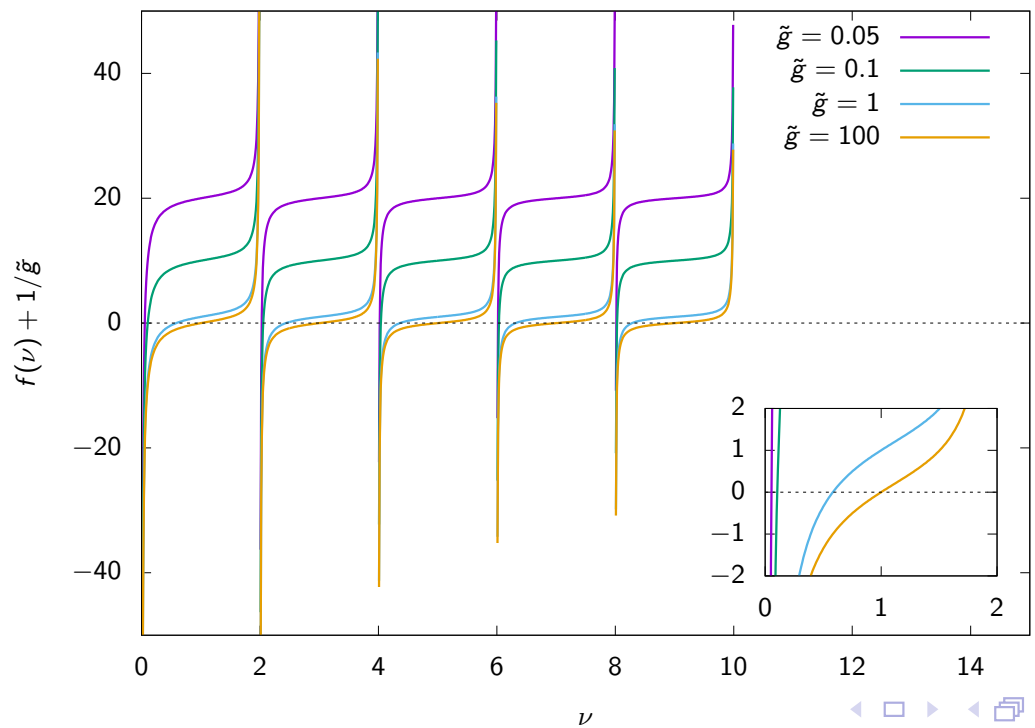
$N = 2$ solution

- The cusp condition then reads:

$$f(\nu) \equiv \frac{\Gamma\left(-\frac{\nu}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)} = -\frac{1}{\tilde{g}}. \tag{30}$$

- The weakly and strongly interacting limits solutions of (30) are

$$\nu(n) = \begin{cases} 2n, & \text{for } \tilde{g} \ll 1 \\ 2n+1, & \text{for } \tilde{g} \gg 1 \end{cases}, \quad \forall\, n \in \mathbb{N}. \tag{31}$$



Asymptotic behavior of the momentum distribution

- From the behavior of the relative motion wave function near $x_i = x_j$ (29),

$$\Psi(x_1, \ldots, x_N) \sim \Psi\left(x_1, \ldots, x_{ij}^{(cm)}, \ldots, x_N\right)\left(1 - \sqrt{2}\frac{|x_{ij}^{(r)}|}{a_{1D}}\right), \quad (32)$$

  it is possible to derive the asymptotic behavior of the momentum distribution

$$n(k) \underset{k\to\infty}{\sim} \frac{2}{\pi a_{1D}^2} k^{-4} \int dx\, \varrho^{(2)}(x, x). \quad (33)$$

- The k^{-4} decay of $n(k)$ at large k is a universal property of δ-like interacting systems, independently of the trapping potential and the dimensionality of the system.
- Contact definition:

$$\mathcal{C} \equiv \lim_{k\to\infty} k^4 n(k). \quad (34)$$

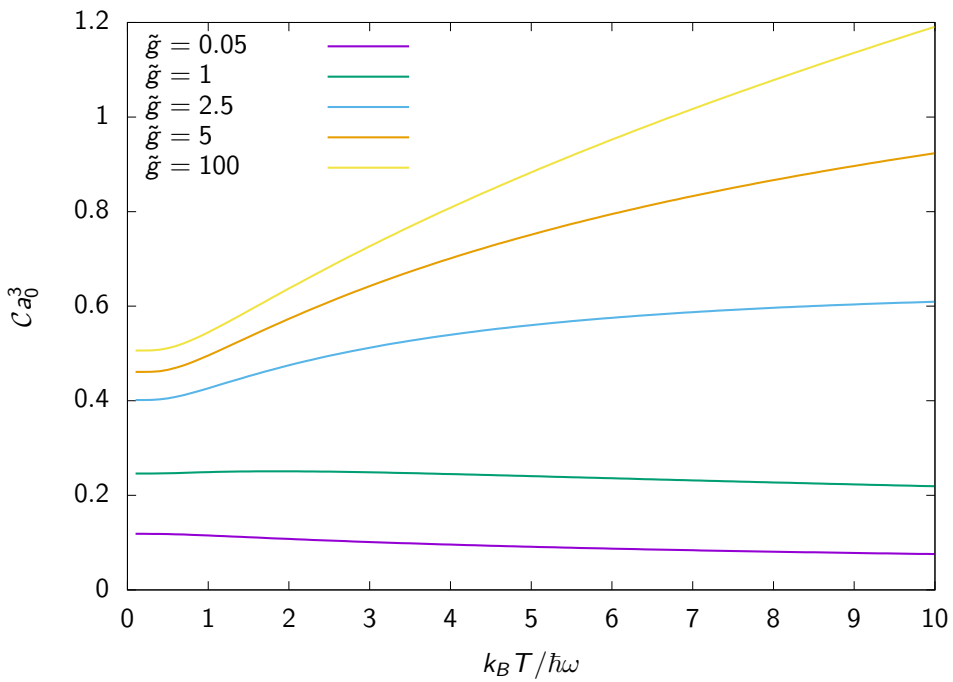
Two-boson contact

- In terms of energy it can be written as

$$\mathcal{C} = -\frac{m^2}{\pi\hbar^4}\frac{\partial E}{\partial g^{-1}}. \tag{35}$$

- For $T > 0$:

$$\mathcal{C}(\tilde{g},\beta) = \frac{2^{5/2}\tilde{g}}{\pi a_0^3 \mathcal{Z}_r}\sum_{\nu} \mathrm{e}^{-\beta\hbar\omega\nu}\left[\psi\left(-\frac{\nu}{2}\right) - \psi\left(-\frac{\nu}{2}+\frac{1}{2}\right)\right]^{-1}. \tag{36}$$



The Tonks-Girardeau limit

- In the strongly interacting scenario $g \to \infty$, the relationship between the many-body wave functions of the bosonic system and a fermionic one is given by [M. Girardeau, *J. Math. Phys.* **1**, 516 (1960)]

$$\Psi_\alpha^{(b)}(x_1, \ldots, x_N) = \Theta(x_1, \ldots, x_N)\Psi_\alpha^{(f)}(x_1, \ldots, x_N), \tag{37}$$

where $\Theta(x_1, \ldots, x_N) \equiv \prod_{i>j} \mathrm{sgn}(x_i - x_j)$ is either $+1$ or -1, in order to compensate the anti-symmetrization of the fermionic wave function

$$\Psi_\alpha^{(f)}(x_1, \ldots, x_N) = (N!)^{-1/2} \det[\phi_{n_i}(x_j)]_{n_i \in \{n_1, \ldots, n_N\};\, x_j \in \{x_1, \ldots, x_N\}}. \tag{38}$$

- The system constituted of strongly interacting bosons can be mapped into the ideal Fermi gas, whose ground-state many-body wave function yields

$$\begin{aligned}\Psi_0^{(b)}(x_1, \ldots, x_N) = {} & \frac{2^{N(N-1)/4}}{a_0^{N/2}} \left(N! \prod_{n=0}^{N-1} n! \sqrt{\pi} \right)^{-1/2} \\ & \times \prod_{i=0}^{N} \mathrm{e}^{-x_i^2/2a_0^2} \prod_{1 \le j < k \le N} |x_k - x_j|.\end{aligned} \tag{39}$$

Finite T density matrix

- The j-body density matrix is given by

$$\varrho^{(j)}(x_1,\ldots,x_j;x_1',\ldots,x_j') = \frac{N!}{(N-j)!}\mathcal{Z}^{-1}\sum_\alpha \mathrm{e}^{-\beta E_\alpha}$$
$$\times \int_{\Re} dx_{j+1}\ldots dx_N \Psi_\alpha^{(b)*}(x_1,\ldots,x_N)\Psi_\alpha^{(b)}(x_1',\ldots,x_j',x_{j+1},\ldots,x_N). \tag{40}$$

- $\varrho^{(1)}$ of the bosonic system in terms of the fermionic wave function, yielding

$$\varrho^{(1)}(x,x') = \frac{N}{\mathcal{Z}}\sum_\alpha \mathrm{e}^{-\beta E_\alpha}\sum_{j=1}^{N-1}\binom{N-1}{j}(-2)^j[\mathrm{sgn}(x-x')]^j$$
$$\times \int_x^{x'} dx_2\ldots dx_{j+1}\int dx_{j+2}\ldots dx_N \Psi_\alpha^{(f)}(x,x_2,\ldots,x_N)\Psi_\alpha^{(f)}(x',x_2,\ldots,x_N). \tag{41}$$

Momentum distribution in the TG limit

- For small distances, it is possible to algebraically work out the last expression and approximate it by

$$\varrho^{(1)}(x,x') \approx \frac{|x'-x|^3}{2} F(x), \tag{42}$$

where

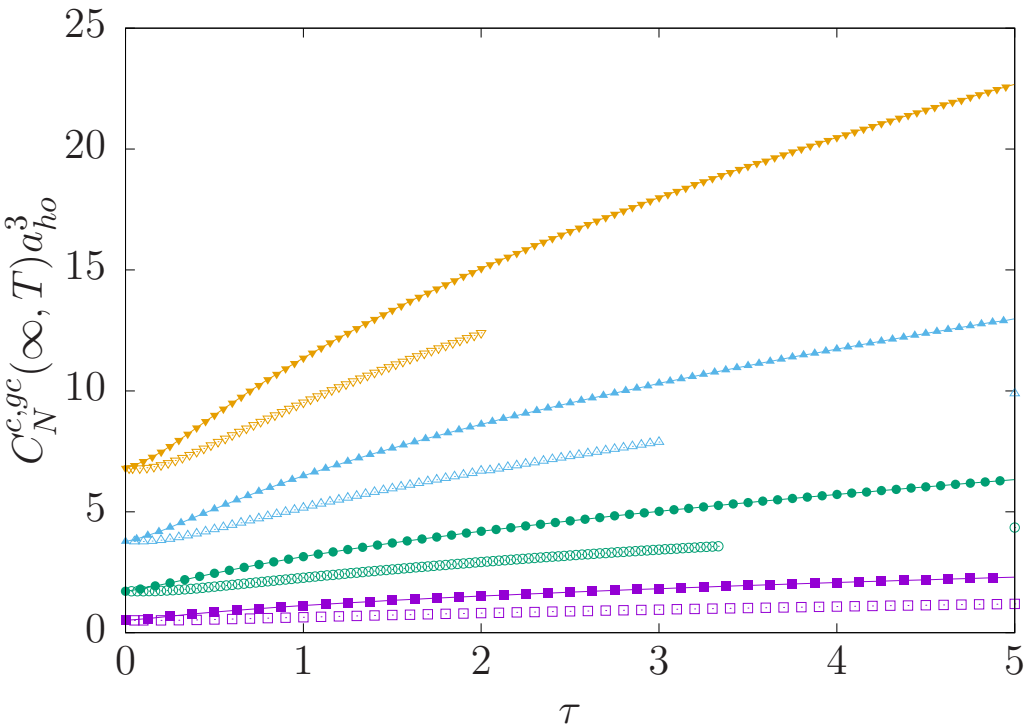
$$F(x) = \mathcal{Z}^{-1} \sum_{n_1,n_2,\ldots,n_N} \mathrm{e}^{-\beta \sum_{i=1}^{N} \epsilon_{n_i}} \times \sum_{j\neq k} \left\{ \left[\phi_{n_j}(x)\partial_x \phi_{n_k}(x)\right]^2 - \phi_{n_j}(x)\phi_{n_k}(x)\partial_x\phi_{n_j}(x)\partial_x\phi_{n_k}(x) \right\} \tag{43}$$

- From the momentum distribution $n(k) = \frac{1}{2\pi}\int dx \int dx' \, \mathrm{e}^{ik(x-x')}\varrho^{(1)}(x,x')$ and making use of the asymptotics of the Fourier transform $\int dx \, \mathrm{e}^{-ik(x-x_0)}|x-x_0|^{a-1} f(x) = \frac{2}{k^a} f(x_0)\cos(\pi a/2)\Gamma(a)$, and the definition of the contact $\mathcal{C} \equiv k^4 n(k)$ as $k \to \infty$ we get

$$\mathcal{C} = \frac{2}{\pi}\int_{-\infty}^{+\infty} dx \, F(x). \tag{44}$$

Tonks-Girardeau Tan's contact

$$\mathcal{C} = \frac{2}{\pi}\int dx\, \mathcal{Z}^{-1} \sum_{n_1,n_2,\ldots,n_N} \mathrm{e}^{-\beta\sum_{i=1}^{N}\epsilon_{n_i}} \times \sum_{j\neq k}\left\{\left[\phi_{n_j}(x)\partial_x\phi_{n_k}(x)\right]^2 - \phi_{n_j}(x)\phi_{n_k}(x)\partial_x\phi_{n_j}(x)\partial_x\phi_{n_k}(x)\right\} \tag{45}$$



arXiv:1908.08714

Zero-temperature scaling

- It was shown in [M. Rizzi *et al.*, *Phys. Rev. A* **98**, 043607 (2018)] that the contact for N bosons can be expressed a function of the two-boson contact $\mathcal{C}_N = \mathcal{C}_N(\mathcal{C}_2)$. Also, it was verified that the scaling relation

$$f_N(\tilde{g}, T = 0) \equiv \frac{\mathcal{C}_N(\tilde{g}, T = 0)}{\mathcal{C}_N(\tilde{g} \to \infty, T = 0)}, \tag{46}$$

  where $\tilde{g} \equiv -a_0 a_{1\mathrm{D}}^{-1}/\sqrt{N}$ and $\mathcal{C}_N(\tilde{g}, T = 0) \propto N^{5/2} - \gamma N^{\eta}$, establishes the equality

$$f_N(\tilde{g}, T = 0) \simeq f_2(\tilde{g}, T = 0). \tag{47}$$

- In particular, in the Tonks-Girardeau limit, $\gamma \approx 1$ and $\eta = 3/4$.

Large-temperature scaling

- For temperatures large enough, $\tau \gg 1$, quantum correlations become negligible in the system, so that the contact for N particles is given by the two-particle contact times the number of pairs,

$$\mathcal{C}_N(\tilde{g}, \tau \gg 1) = \frac{N(N-1)}{2}\mathcal{C}_2(\tilde{g}, \tau \gg 1). \tag{48}$$

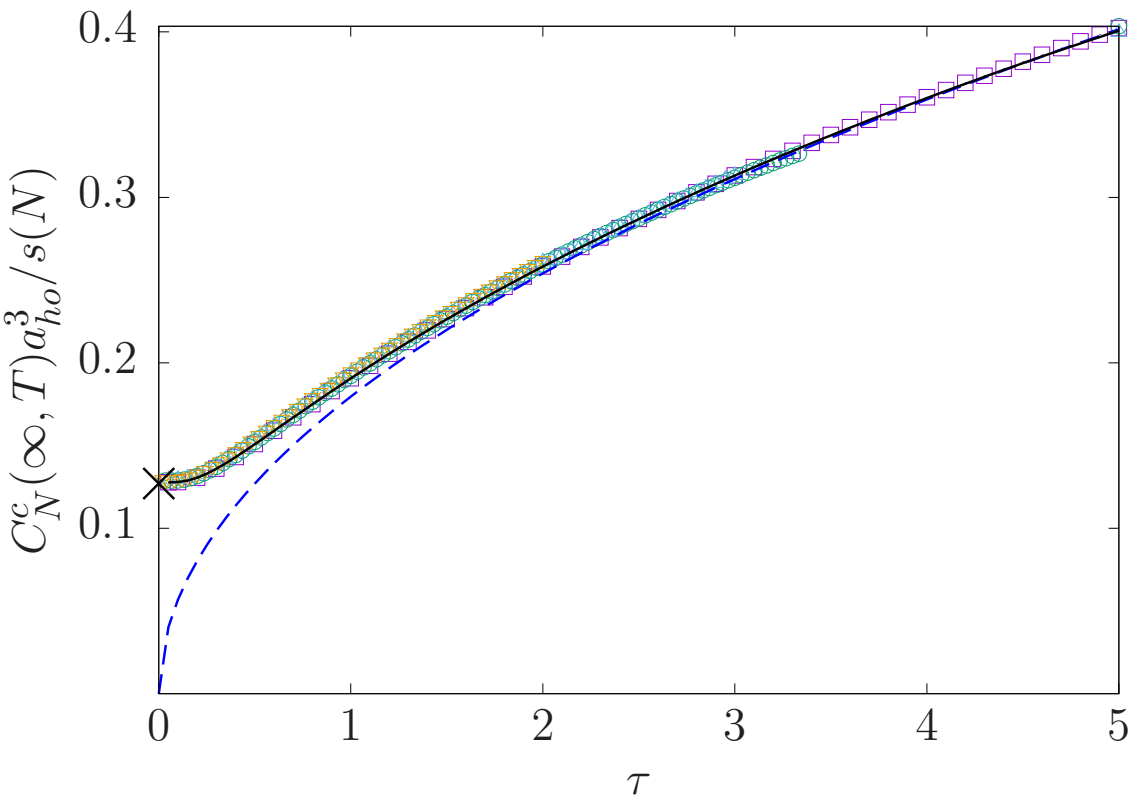
- Then, in the strongly interacting limit we find can extract all N dependence as

$$\mathcal{C}_N\left(\tilde{g} \to \infty, \tau \gg 1\right) = \frac{\sqrt{\tau}}{a_0^3 \pi^{3/2}}\left(N^{5/2} - N^{3/2}\right). \tag{49}$$

Generalized scaling conjecture in the TG limit

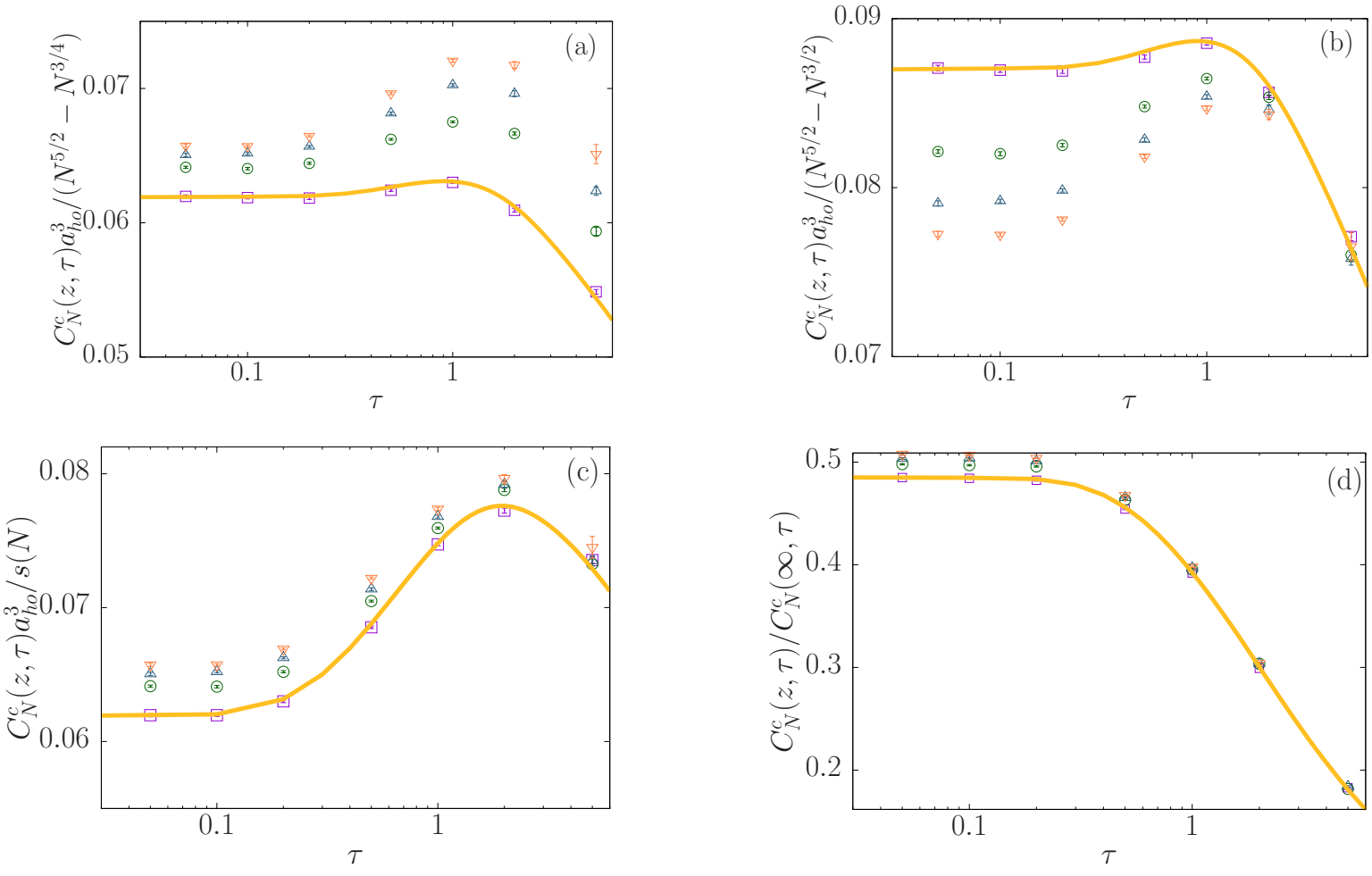
- Zero-temperature scaling: $\mathcal{C}_N(\tilde{g} \to \infty, \tau \ll 1) \propto N^{5/2} - N^{3/4}$
- Large-temperature scaling: $\mathcal{C}_N(\tilde{g} \to \infty, \tau \gg 1) \propto N^{5/2} - N^{3/2}$
- Generalized scaling conjecture:

$$\mathcal{C}_N(\tilde{g} \to \infty, \tau) \propto N^{5/2} - N^{3/4[1+\exp(-2/\tau)]} \tag{50}$$



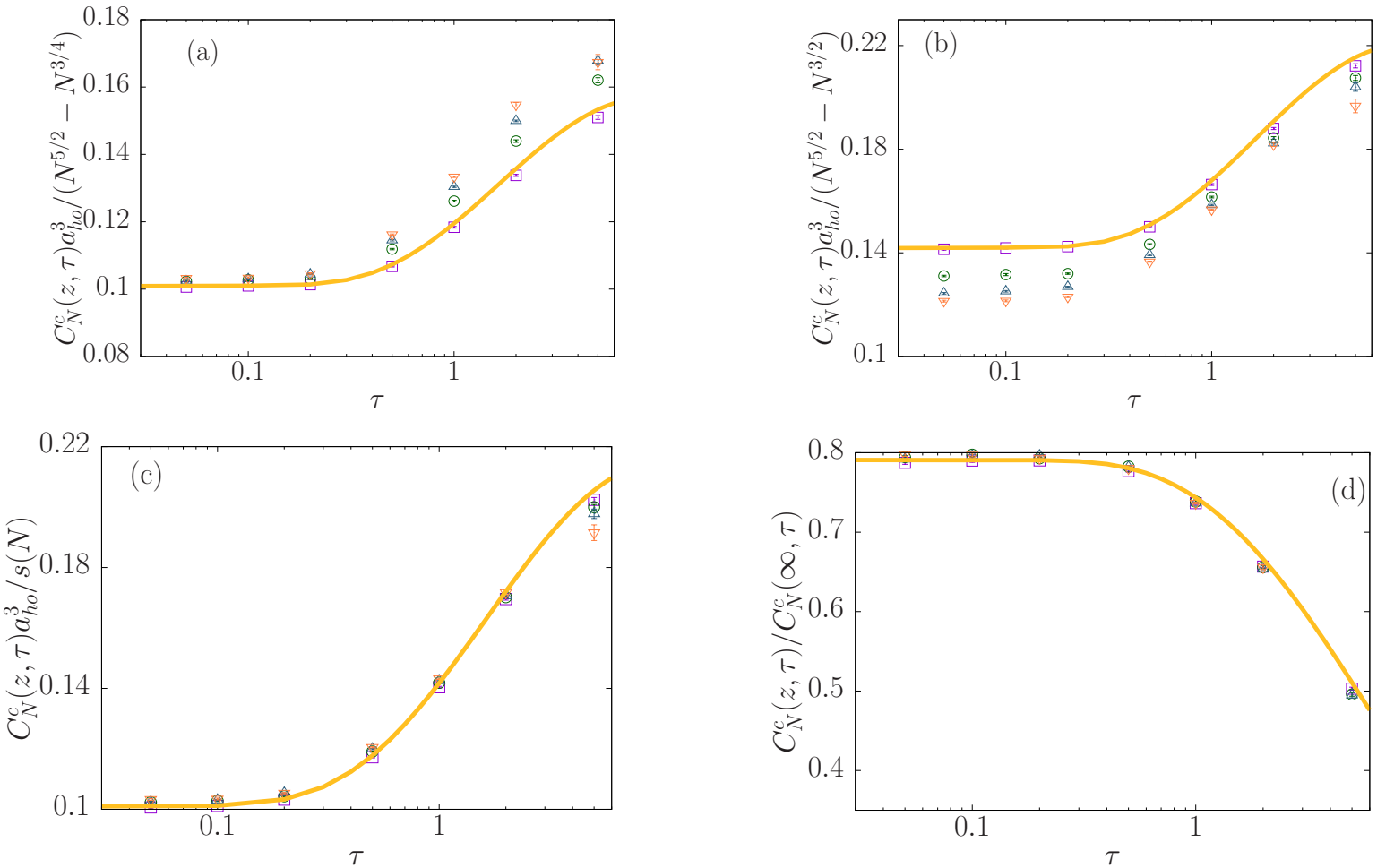
arXiv:1908.08714

Intermediate-interaction scaling: $\tilde{g} = 1$



arXiv:1908.08714

Intermediate-interaction scaling: $\tilde{g} = 2.5$



arXiv:1908.08714

Conclusive remarks

- We evaluated the contact for the harmonically trapped Tonks-Girardeau gas
- We worked out a scaling factor for the contact in the TG limit for any temperature.
- Relying on QMC simulations, we showed that all previously stated scalings in the TG limit also apply for the finite interaction regime.
- Summary: The contact rescaled by the generalized scaling function and the ratio between the finite-interaction contact and its correspondent in the Tonks-Girardeau limit are both universal functions of $\tilde{g}$ and τ.

Thank you!