QFT notes

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1 Preliminaries

Metric tensor:

$$g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \tag{1}$$

Satisfying

$$g_{\mu\nu} g^{\nu\rho} = \delta^{\rho}_{\mu} \tag{2}$$

The relativistic line is given by

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = (dx^0)^2 - d\vec{x} \cdot d\vec{x}, \tag{3}$$

with

$$x^{\mu} = (x^0, \vec{x}), \quad x_{\mu} = g_{\mu\nu} x^{\nu} = (x^0, -\vec{x}) \tag{4}$$

$$p \cdot x = g_{\mu\nu} p^{\mu} x^{\nu} = p^0 x^0 - \vec{p} \cdot \vec{x} \tag{5}$$

Massive particle:

$$p^2 = p^{\mu} p_{\mu} = E^2 - |\vec{p}|^2 = m^2, \tag{6}$$

where the 4-momentum is defined as

$$p^{\mu} = m \frac{dx^{\mu}}{ds} \tag{7}$$

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right) \tag{8}$$

2 Classical field theory

2.1 Lagrangian formalism

Let a Lagrangian density be the function of a field ϕ and its derivative $\partial_m u\phi$. Then, the action is given by

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\phi, \partial_{\mu}\phi). \tag{9}$$

The principle of least action states that whenever a system evolves from one configuration to another between times t_1 and t_2, it does so along a path in which S is minimum:

$$\begin{aligned}
\delta S &= 0 && (10)\\
\delta S &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta(\partial_{\mu}\phi) \right] && (11)\\
&= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \right) \delta\phi + \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta\phi \right) \right]. && (12)
\end{aligned}$$

Note that the last term is a total derivative and it vanishes for any $\delta\phi$ (it can be regarded as a surface integral over the boundary of the $4d$ space-time). Since the initial and final field configurations are assumed to be given, $\delta\phi$ is zero at both the temporal beginning and end of such a region. If we restrict ourselves to deformations $\delta\phi$ that vanish on the spatial boundary as well, then the surface term is zero. Therefore, and requiring $\delta\phi = 0$ for all such paths, we have the Euler-Lagrange equation of motion for the field,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{13}$$

2.2 Lorentz invariance

Consider a Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{14}$$

where Λ^μ_ν satisfies

$$\Lambda^\mu_\sigma g^{\sigma\tau} \Lambda^\nu_\tau = g^{\mu\nu}. \tag{15}$$

As an example, consider a rotation by θ around the x^3-axis and a boost by v along the x^1-axis. Their respective Lorentz transformations are

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

with $\gamma = (1 - v^2)^{-1/2}$.

Now let's consider a Lorentz transformation $x \to \Lambda x$, so that the field transforms as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x). \tag{17}$$

If the field $\phi(x)$ solves the equations of motion of a Lorentz invariant theory, then $\phi(\Lambda^{-1}x)$ also does. We can ensure that such a property holds by requiring that the action is Lorentz invariant.

Example: Klein-Gordon equation

Consider a real scalar field transforming as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$, which implies that the derivative of the field transform as

$$(\partial_\mu \phi)(x) \to (\Lambda^{-1})^\nu_\mu (\partial_\nu \phi) y, \tag{18}$$

where $y \equiv \Lambda^{-1} x$. This means that the derivative term in the Lagrangian transforms as

$$\mathcal{L}(x) = \partial_\mu \phi(x) \partial^\nu \phi(x) \to (\Lambda^{-1})^\rho_\mu (\partial_\rho \phi)(y) (\Lambda^{-1})^\nu_\sigma (\partial^\sigma \phi)(y) \tag{19}$$

$$= (\partial_\rho \phi)(y)(\partial^\nu \phi)(y) = \mathcal{L}(y). \tag{20}$$

The potential terms transform in the same way $\phi(x)^2 \to \phi(y)^2$. Putting all of them together, we find that the action is invariant under a Lorentz transformation:

$$S = \int d^4x\, \mathcal{L}(x) \to \int d^4y\, \mathcal{L}(y) = S. \tag{21}$$

2.3 Noether's theorem

Every continuous symmetry of the Lagrangian gives rise to a conserved current $j^\mu(x)$ such that

$$\partial_\mu j^\mu = 0 \qquad \partial_t j^0 + \nabla \cdot \vec{j} = 0 \tag{22}$$

A conserved current implies a conversed charge Q defined as

$$Q = \int_{\mathbb{R}}^3 j^0, \tag{23}$$

which we can see that

$$\frac{dQ}{dt} = \int_{\mathbb{R}^3} d^3x \partial_t j^0 = -\int d^3x \nabla \cdot \vec{j} = 0 \tag{24}$$

assuming that $j \to 0$ as $x \to \infty$.

Whenever we have a continuous symmetry we may work infinitesimally, we consider a transformation

$$\delta\phi(x) = X(\phi), \tag{25}$$

is a symmetry if the Lagrangian changes by a derivative $\delta\mathcal{L} = \partial_\mu F^\mu$, for some function $F^\mu(\phi)$. Then, the Lagrangian variation reads

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \tag{26}$$

$$= \left[\partial_\phi\mathcal{L} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right), \tag{27}$$

$$\partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi) = \partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)})\delta\phi + \frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \tag{28}$$

when the eq. of motion are satisfied, we are left with

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi)\right) = \partial_\mu F^\mu(\phi) \tag{29}$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi) - F^\mu(\phi)\right) = 0 \tag{30}$$

$$\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi) - F^\mu(\phi)\right) = j^\mu \tag{31}$$

Example Consider an infinitesimal translation $x^\nu \to x^\nu - \epsilon^\nu$. For the field it means that $\phi(x) \to \phi(x) + \epsilon^\nu\partial_\nu\phi(x)$

The Lagrangian transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \epsilon^\nu\partial_\nu\mathcal{L}(x). \tag{32}$$

Since the change in the Lagrangian is a total derivative, we may invoke Noether's theorem which will provide four conserved currents $(j^\mu)_\nu$, one for each of the translations, $\nu = 0, 1, 2, 3$.

$$(j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \equiv T^{\mu\nu}, \tag{33}$$

is called the energy-momentum tensor, which satisfies

$$\partial_\mu T^{\mu\nu} = 0. \tag{34}$$

The four converse quantities are

$$E = \int d^3x T^{00}, \quad P^i = \int d^3x T^{0i}. \tag{35}$$

Example of the Energy-momentum tensor

Consider the simple scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{36}$$

then

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \tag{37}$$

$$= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \delta^\mu_\nu \mathcal{L} \tag{38}$$

$$T^{\mu\nu} = g^{\sigma\nu} T^\mu_\nu = \frac{1}{2} \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}. \tag{39}$$

From the eq. of motion it is possible to verify that $\partial_\mu T^{\mu\nu} = 0$, which give rise the following conserved quantities

$$E = \frac{1}{2} \int d^3x \dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2, \qquad P = \int d^3x \dot{\phi} \partial^i \phi. \tag{40}$$

2.4 Hamiltonian formulation

The Lagrangian formulation of field theory is particularly suited for relativistic dynamics because all expressions are Lorentz invariant. Hamiltonian formulation provides an easy path towards the transition to quantum mechanics.

Recall that for a system we can define the conjugate momentum

$$p \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}}, \tag{41}$$

where

$$\dot{q} = \partial_t q, \tag{42}$$

for each dynamical variable q. The Hamiltonian is

$$H = \sum p\dot{q} - L. \tag{43}$$

Generalization for a field:

$$\Pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}, \tag{44}$$

momentum density conjugate of the field $\phi(x)$. The Hamiltonian reads

$$H = \int d^3x (\Pi(x)\dot{\phi}(x) - \mathcal{L}). \tag{45}$$

Consider our simple scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \partial^\mu \phi - \frac{1}{2} m^2 \phi^2, \tag{46}$$

we have that $\Pi(x) = \dot{\phi}$, and thence the Hamiltonian is given by

$$H = \frac{1}{2} \int d^3x (\dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2). \tag{47}$$

3 Canonical quantization

3.1 The Klein-Gordon field as harmonic oscillators

We consider a real Klein-Gordon field, and in order to quantize it, we promote ϕ and Π to operators, and we impose suitable commutation relations:

$$[q_i, p_j] = i\delta_{ij}, \qquad [q_i, q_j] = [p_i, p_j] = 0. \tag{48}$$

For a continuous system the generalization is quite natural, since $\Pi(\vec{x})$ is the momentum density, we get a Dirac delta instead of a Kronecker delta

$$[phi(x), \Pi(y)] = i\delta^{(3)}(x-y), \qquad [\phi(x), \phi(y)] = [\Pi(x), \Pi(y)] = 0. \tag{49}$$

The Hamiltonian, which is a function of ϕ and Π, also becomes an operator. In order to find the spectrum from the Hamiltonian, let's begin by writing the field in Fourier space:

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} e^{ipx} \phi(p,t). \tag{50}$$

The Klein-Gordon equation becomes

$$(\partial_t^2 + p^2 + m^2)\phi(p,t) = 0. \tag{51}$$

The above equation is the eq. of motion for a harmonic oscillator with frequency $\omega_p^2 = p^2 + m^2$. Let's recall how to find the spectrum of HO. We firstly define the ladder operators

$$\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger), \qquad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger), \tag{52}$$

where the Hamiltonian is

$$H = p^2/2 + \omega^2\phi^2/2. \tag{53}$$

From $[\phi, p] = i$, we have that $[a, a^\dagger] = 1$. With these relations, we can rewrite the Hamiltonian as

$$H = \omega(a^\dagger a + 1/2). \tag{54}$$

The zero-point state $|0\rangle$ is such that $a|0\rangle = 0$ is an eigenstate of H with eigenvalue $\omega/2$. Furthermore, with the commutators $[H, a^\dagger] = \omega a^\dagger$, $[H, a], -\omega a$, it is possible to verify that the states

$$|n\rangle \equiv (a^\dagger)|0\rangle, \tag{55}$$

they are eigenstates of H with eigenvalues $(n + 1/2)\omega$.

With that, we can proceed to inspect the spectrum of the KG Hamiltonian within an analogous formulation. Note: each Fourier mode of the field will be treated as an independent oscillator, with its own a and $a^\dagger$. Analogously we have that

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{ipx} + a^\dagger e^{-ipx}), \tag{56}$$

$$\Pi(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{w_p}{2}} (a_p e^{ipx} - a^\dagger e^{-ipx}). \tag{57}$$

One can show that the commutation relation $[a, a^\dagger] = 1$ becomes $[a_p, a^\dagger q] = (2\pi)^3\delta^{(3)}(p-q)$, and that

$$[\phi(x), \Pi(y)] = -\frac{i}{2} \int \frac{d^3p d^3q}{(2\pi)^6} \sqrt{\frac{\omega_q}{\omega_p}} ([a^\dagger_{-p}, a_q] - [a_p, a^\dagger_- q]) e^{i(px+qy)} = i\delta^{(3)}(x-y). \tag{58}$$

With those, we can express the KG Hamiltonian in terms of the ladder operators. Starting with

$$H = \frac{1}{2} \int d^3x (\Pi^2 + (\nabla\phi)^2 + m^2\phi^2), \tag{59}$$

and substituting the expressions for the operators, it yields

$$H = \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} e^{i(p+q)x} \left[-\frac{\sqrt{\omega_p \omega_q}}{4} (a_p - a^\dagger_{-p})(a_q - a \dagger -q) + \frac{m^2 - \vec{p}\cdot\vec{q}}{4\sqrt{\omega_p\omega_q}} (a_p + a^\dagger_{-p})(a_q + a\dagger_{-q}) \right] \tag{60}$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p] \right), \tag{61}$$

where $[a_p, a_p^\dagger] = (2\pi^3)\delta^{(3)}(0)$. We observe that in the above expression the second term if proportional to $\delta^{(3)}(0)$, which is infinite. It is regarded as the sum over all modes of zero-point energies. Although, this infinite term appears in the equations, it does not have any implication in real life, once in experiments there are no infinite measures, it is only possible to measure the difference between energy levels and the ground state. Let's take a closer look at the vacuum $a_p|0\rangle = 0$. Then, the ground state energy comes from the second term

$$H|0\rangle = E_0|0\rangle = \left(\int d^3p\,\omega_p\,\delta^{(3)}(0)\right)|0\rangle = \infty|0\rangle. \tag{62}$$

This infinite only appears because space is infinitely large (infra-red divergence). Instead of that, we can consider inserting our theory inside a volume V,

$$(2\pi)^3\delta^{(3)}(0) = \lim_{V\to\infty}\int_V d^3x\,e^{ixp}\Big|_{p=0} = V. \tag{63}$$

Instead of calculating the total energy, we can evaluate the energy density

$$\frac{E_0}{V} = \frac{1}{2}\int\frac{d^3x}{(2\pi)^3}\omega_p, \tag{64}$$

observe that this term is still infinite, it is recognized as the sum of the ground state energies for each HO. But as $p\to\infty$ $E_0/V\to\infty$, this is a high-frequency infinity known as *ultra-violet divergence*.

In a more practical way, we can redefine the Hamiltonian by subtracting the problematic term:

$$H = \int\frac{d^3x}{(2\pi)^3}\omega_{\vec{p}}a_{\vec{p}}^\dagger a_{\vec{p}}, \tag{65}$$

so that

$$H|0\rangle = 0. \tag{66}$$

Example: The Camisir effect Consider a box of size L and insert two plates separated separated by a distance $d \ll L$ in the x-direction. The plates impose $\phi(x) = 0$ at the position of the plates, and periodic boundary conditions such that $\phi(\vec{x}) = \phi(\vec{x}+L\vec{n})$, where $\vec{n} = (1,0,0)$. The momentum of the field inside the plates is quantized as

$$\vec{p} = (n\pi/d, p_y, p_z), n\in\mathbb{Z}. \tag{67}$$

We want to evaluate the ground state energy between the plates for a scalr field:

$$E(d) = \sum_{n=1}^{\infty}\int dp_y\,dp_z\sqrt{\left(\frac{n\pi}{2}\right)^2 + p_y^2 + p_z^2} \tag{68}$$

We see that E is infinite, because it comes from arbitrarily high momentum modes. This shouldn't reflect real occurrences in experiments. Mathematically we want to consider only modes with momentum $p \gg 1/a$, for some length scale $a \ll d$, known as the ultraviolet cut-off. One way of doing this is inserting an exponential decaying term into the eqs.

$$E(d) = \sum_{n=1}^{\infty}\int dp_y dp_z\sqrt{\left(\frac{n\pi}{2}\right)^2 + p_y^2 + p_z^2}e^{-a((n\pi/d)^2+p_y^2+p_z^2)^{1/2}}, \tag{69}$$

which for $a\to 0$ we simply recover the full infinite expression. In order to simplify things, let's consider 1d case:

$$E(d) = \frac{\pi}{2d}\sum_{n=1}^{\infty}n \to \frac{\pi}{2d}\sum_{n=1}^{\infty}ne^{-an\pi/d} \tag{70}$$

$$= -\frac{1}{2}\partial_a\sum_n e^{-an\pi/d} = -\frac{1}{2}\partial_a\left(\frac{1}{1-e^{a\pi/d}}\right) \tag{71}$$

$$= \frac{\pi}{2d}\frac{e^{a\pi/d}}{(e^{a\pi/d}-1)^2} \tag{72}$$

Considering $a \ll d$, we have that

$$E(d) = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + \mathcal{O}(a^2). \tag{73}$$

The full energy inside the box is calculated as

$$E(d) + E(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24}\left(\frac{1}{d} - \frac{1}{L-d}\right) + \mathcal{O}(a^2). \tag{74}$$

The force is provided by

$$\partial_d E = \frac{\pi}{24d^2} + \cdots, \tag{75}$$

The important point is that as $a \to 0$, and $L \to \infty$, the force remains finite.

Coming back to the spectrum of the theory all other energy eigenstates can be built by acting with the creation operator over the vacuum. Arbitrary state

$$a_p^\dagger a_q^\dagger \cdots |0\rangle, \tag{76}$$

which is an eigenstate of the Hamiltonian with energy $\omega_p + \omega_q + \cdots$. Let's do some interpretation for these states.

From

$$P^i = \int T^{0i} d^3x = \int \Pi \partial_i \phi d^3x, \tag{77}$$

total momentum operator of the field. It reads

$$\vec{P} = -\int d^3x \Pi(x) \nabla\phi(x) = \int \frac{d^3x}{(2\pi)^3} \vec{p} a_p^\dagger a_p. \tag{78}$$

The operator $a_p^\dagger$ creates momentum $\vec{p}$ and energy $\omega_p = \sqrt{p^2 + m^2}$. Similarly, the state (76) has momentum $p + q + \cdots$.

As $[a_p^\dagger, a_q^\dagger] = 0$, we have that that $a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle$, which means interchange of particles without any cost/sign. Therefore, we conclude that the KG field obeys Bose-Einstein statistics.

Now, let's consider the interpretation of the state $\phi(x)|0\rangle$. From (56), we have that

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |p\rangle, \tag{79}$$

and that

$$\langle 0|\phi(x)|p\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p'}} (a_{p'} e^{ip'x} + \text{c.c.}) \sqrt{2E_p} a_p^\dagger |0\rangle = e^{ipx}. \tag{80}$$

This means that the operator, acting over the vacuum, creates a particle at position x. Also, it is possible to consider this as the position-space representation of a single particle wavefunction of the state $|p\rangle$, similarly to nonrelativistic QM we have that $\langle x|p\rangle \propto e^{ipx}$ is the wavefunction of the state $|p\rangle$.

Let us consider the issue of proper, relativistic normalization. We define the vacuum normalization as $\langle 0|0\rangle = 1$, and then the one particle state $|\vec{p}\rangle = a_{\vec{p}}^\dagger |0\rangle$, satisfying

$$\langle \vec{p}|\vec{q}\rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}). \tag{81}$$

Question: Is this quantity Lorentz invariant?

Let's consider the Lorentz transformation

$$p^\mu \to (p')^\mu = \Lambda^\mu_\nu p^\nu, \tag{82}$$

such that the 3-vector transforms as $\vec{p} \to \vec{p'}$. In QM, we like relating states by a unitary transformation

$$|\vec{p}\rangle \to |\vec{p'}\rangle = U(\Lambda)|\vec{p}\rangle. \tag{83}$$

This would imply that the normalization of $|\vec{p}\rangle$ and $|\vec{p'}\rangle$ are the same whenever they are related by a Lorentz transformation. In general we have that $|\vec{p}\rangle \to |\vec{p'}\rangle = \lambda(\vec{p}, \vec{p'})|\vec{p'}\rangle$, for some unknown function λ. On the other hand, let us take a closer look at a quantity that we know that is Lorentz invariant.

Consider the identity operator

$$1 = \int \frac{d^3p}{(2\pi)^3} |\vec{p}\rangle\langle\vec{p}| \tag{84}$$

Question: are $\frac{d^3p}{(2\pi)^3}$ and $|\vec{p}\rangle\langle\vec{p}|$ Lorentz invariant?

Claim: The Lorentz invariant measure is given by

$$\int \frac{d^3p}{2E_p}. \tag{85}$$

Proof: As $\int d^4p$ is Lorentz invariant, the relativistic dispersion relation for a massive particle $p_\mu p^\mu = m^2 \to E_p^2 = p^2 + m^2 = p_0$ is also Lorentz invariant. Because of that, the following combination must be Lorentz invariant

$$\int d^4x \delta(p_0^2 - \vec{p}^2 - m^2) = \int \frac{d^3p}{2p_0}. \tag{86}$$

Now, the Lorentz invariant δ-function for 3-vectors is $2E_{\vec{p}}\delta^{(3)}(\vec{p} - \vec{q})$, because

$$\int \frac{d^3p}{2E_p} 2E_p \delta^{(3)}(\vec{p} - \vec{q}) = 1 \tag{87}$$

Finally, we have that the relativistically normalized momentum states are

$$|p\rangle \equiv \sqrt{2E_p}|\vec{p}\rangle = \sqrt{2E_p} a_{\vec{p}}^\dagger |0\rangle. \tag{88}$$

The state $|p\rangle$ differs from $|\vec{p}\rangle$ by a factor of $\sqrt{2E_p}$. They satisfy

$$\langle p|q\rangle = (2\pi)^3 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q}), \tag{89}$$

and the identify on one-particle state

$$1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |p\rangle\langle q|. \tag{90}$$

3.2 Complex scalar fields

Let us consider a Lagrangian of a complex scalar field $\Psi(x)$

$$\mathcal{L} = \partial_\mu \Psi^* \partial^\mu \Psi - M^2 \Psi^* \Psi. \tag{91}$$

Is worth to note that we could write the complex field in terms of real scalar ones as $\Psi = (\phi_1 + i\phi_2)/\sqrt{2}$, and then we could recover the Lagrangian for a real scalar field. For the complex scalar field, the eqs. of motion are

$$\partial_\mu \partial^\mu \Psi + M^2 \Psi = 0 \tag{92}$$

$$\partial_\mu \partial^\mu \Psi^* + M^2 \Psi^* = 0. \tag{93}$$

The analogous expressions for field operator reads

$$\Psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(b_p e^{ipx} + c_p^\dagger e^{-ipx}\right) \tag{94}$$

$$\Psi^* = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(b_p^\dagger e^{-ipx} + c_p e^{ipx}\right) \tag{95}$$

This follows because, since the classical field is not real, its quantum correspondent is not hermitian, that is the reason why we have two different operators b and c in the above expressions. Recalling that the classical field momentum is $\Pi = \partial_{\dot{\Psi}}\mathcal{L} = \dot{\Psi}^*$, its quantum version is

$$\Pi = i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(b_p^\dagger e^{-ipx} - c_p e^{ipx}\right) \tag{96}$$

$$\Pi^\dagger = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(b_p e^{ipx} - c_p^\dagger e^{-ipx}\right), \tag{97}$$

with commutation relations given by $[\Psi(x), \Pi(y)] = i\delta^{(3)}(x-y)$ and all the other usual commutation relations. From those, one can derive the commutation relations for the operators b and c:

$$[b_p, b_q^\dagger] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{98}$$

$$[c_p, c_q^\dagger] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}), \tag{99}$$

and all the usual others that vanish.

The quantization of a complex scalar field gives rise to two different operators, They can interpreted as creating two types of particle with same mass: particles and anti-particles. On the other hand, a real scalar field allow for only one kind of particle: meaning that the particle is its own anti-particle. Recalling that the theory has a classical conversed charge

$$Q = i\int d^3x(\dot{\Psi}^*\Psi - \Psi^*\dot{\Psi}) = i\int d^3x(\Pi\Psi - \Psi^*\Pi^*), \tag{100}$$

which becomes

$$Q = \int \frac{d^3p}{(2\pi)^3}(c_p^\dagger c_p - b_p^\dagger b_p) = N_c - N_b, \tag{101}$$

which means that Q represents the difference between particles and antiparticles, and $[H, Q] = 0$, ensuring that Q is conversed.

3.2.1 The Klein-Gordon field in space and time

Here we will consider the KG field in the Heisenberg picture, which will provide us with a better framework to deal with time-dependent quantities. In the Heisenberg pictures, the operators evolve in time as

$$\mathcal{O} = e^{iHt}\mathcal{O}e^{-iHt}, \tag{102}$$

obeying the eq. of motion

$$i\partial_t\mathcal{O} = [\mathcal{O}, H]. \tag{103}$$

The Heisenberg eqs. of motion of ϕ and Π read

$$i\partial_t\phi(x,t) = [\phi(x,t), 1/2\int d^3x'(\Pi^2(x',t) + (\nabla\phi(x\,t))^2 + m^2\phi^2(x',t))] \tag{104}$$

$$= \int d^3x' i\delta^{(3)}(x-x')\Pi(x',t) = i\Pi(x,t); \tag{105}$$

$$i\partial_t\Pi(x,t) = [\Pi(x,t), 1/2\int d^3x'(\Pi^2(x',t) + (\nabla\phi(x\,t))^2 + m^2\phi^2(x',t))] \tag{106}$$

$$= \int d^3x'(-i\delta^{(3)}(x-x')(m^2-\nabla^2)\phi(x',t)) = -i(m^2-\nabla^2)\phi(x,t). \tag{107}$$

$$\tag{108}$$

By making using of both eqs. of motion, we simply recover the KG equation

$$\partial_t^2\phi = (\nabla^2 - m^2)\phi. \tag{109}$$

Now, by using the commutation relations $[H, a_{\mathbf{p}}] = -E_{\mathbf{p}}a_{\mathbf{p}}$ and $[H, a_{\mathbf{p}}^\dagger] = E_{\mathbf{p}}a_{\mathbf{p}}^\dagger$. We have that a and $a^\dagger$ follow the time evolution

$$e^{iHt}a_{\mathbf{p}}e^{-iHt} = e^{-iE_{\mathbf{p}}t}a_{\mathbf{p}} \tag{110}$$

$$e^{iHt}a_{\mathbf{p}}^\dagger e^{-iHt} = e^{iE_{\mathbf{p}}t}a_{\mathbf{p}}^\dagger. \tag{111}$$

From the expression for the operator $\phi(x,t)$ and its Heisenberg time evolution, we have that

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}), \tag{112}$$

$p\cdot x = E_{\vec{p}}t - \vec{p}\vec{x}$. It is possible to check that $\phi(x,t)$ satisfies the KG equation. Likewise, $\Pi(x,t)$ can be derived from $\Pi(x,t) = \partial_t\phi(x,t)$. In analogy to (113), it is possible to write

$$e^{-i\vec{P}.\vec{x}}a_{\vec{p}}e^{i\vec{P}.\vec{x}} = e^{i\vec{p}.\vec{x}}a_{\vec{p}} \tag{113}$$

$$e^{i\vec{P}.\vec{x}}a_{\vec{p}}^\dagger e^{-i\vec{P}.\vec{x}} = e^{-i\vec{p}.\vec{x}}a_{\vec{p}}\dagger, \tag{114}$$

and thence $\phi(x) = e^{iPx}\phi(0)e^{-iPx}$, where $Px = Ht - \vec{p}\vec{x}$. Note on notation: $\vec{P}$ is the momentum operator, whose eigenvalue is the total momentum of the system. Whereas, $\vec{p}$ is the momentum of a single Fourier mode of the field, that can be interpreted as the momentum of a single particle in the respective mode.

4 Causality

Consider the amplitude for a free particle to propagate from x_0 to x:

$$U(t) = \langle x|e^{-iHt}|x_0\rangle. \tag{115}$$

Nonrelativistic case: $E = p^2/2m$.

$$U(t) = \langle x|e^{-itp^2/2m}|x_0\rangle \tag{116}$$

$$= \int d^3p\langle x|e^{-itp^2/2m}|p\rangle\langle p|x_0\rangle \tag{117}$$

$$= \int \frac{d^3p}{(2\pi)^3}e^{-ip^2/2mt}e^{ip(x-x_0)} \tag{118}$$

$$= \left(\frac{m}{2\pi it}\right)^{3/2} e^{im(x-x_0)^2/2t}. \tag{119}$$

Note that for any x and t, $U(t) \neq 0$. First thought: this may violate causality.

Relativistic case: $E^2 = p^2 + m^2$

$$
\begin{aligned}
U(t) &= \langle x| e^{-it\sqrt{p^2+m^2}} |x_0\rangle &(120)\\
&= \int d^3p \langle x| e^{-it\sqrt{p^2+m^2}} |p\rangle\langle p|x_0\rangle &(121)\\
&= \int \frac{d^3p}{(2\pi)^3} e^{-it\sqrt{p^2+m^2}} e^{ip(x-x_0)} &(122)\\
&= \frac{1}{2\pi^2(x-x_0)} \int_0^\infty dp\, p \sin(p|x-x_0|) e^{-it\sqrt{p^2+m^2}}. &(123)
\end{aligned}
$$

The integral can be evaluated in terms of Bessel functions. For the sake of the discussion, let's simply analyze its asymptotic behavior at $x^2 \gg t^2$ (outside the light-cone). The function $px - t\sqrt{p^2+m^2}$ has a stationary point at $p = imx/\sqrt{x^2-t^2}$. Plugging this value for p in the time evolution we have that

$$
U(t) \sim e^{-m\sqrt{x^2-t^2}}. \tag{124}
$$

The propagation amplitude is nonzero outside the light-cone, thence causality if violated. Let's see how QFT solves this issue.

Let's define the amplitude for a particle to propagate from y to x

$$
\langle 0|\phi(x)\phi(y)|0\rangle \equiv D(x-y). \tag{125}
$$

The only term from the ϕ operator that counts for this is $\langle 0|a_{\vec{p}} a^\dagger_{\vec{q}}|0\rangle = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$, so that

$$
D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}. \tag{126}
$$

Let's evaluate the integral for different cases.

1. Time-like:

$$
\begin{cases} x^0 - y^0 = t \\ \vec{x} - \vec{y} = 0 \end{cases} \tag{127}
$$

$$
D(x-y) = \frac{1}{2\pi^2} \int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}} e^{-it\sqrt{p^2+m^2}} \tag{128}
$$

$$
= \frac{1}{4\pi^2} \int_m^\infty dE\, \sqrt{E^2-m^2} e^{-iEt} \tag{129}
$$

$$
\sim_{t\to\infty} e^{-imt}. \tag{130}
$$

2. Space-like:

$$
\begin{cases} x^0 - y^0 = 0 \\ \vec{x} - \vec{y} = \vec{r} \end{cases} \tag{131}
$$

$$
\begin{aligned}
D(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}.\vec{r}} &(132)\\
&= \frac{1}{4\pi^2} \int_0^\infty dp\, \frac{p^2}{2E_p} \frac{e^{ipr} - e^{-ipr}}{ipr} &(133)\\
&= -\frac{i}{8\pi^2 r} \int_{-\infty}^{+\infty} dp \frac{p e^{ipr}}{\sqrt{p^2+m^2}}. &(134)
\end{aligned}
$$

The integrand has branch cuts at $\pm im$. Because of that we define $\rho \equiv -ip$, thus

$$D(x-y) = \frac{1}{4\pi^2 r}\int_m^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{m^2-\rho^2}} \sim_{r\to\infty} e^{-mr}. \tag{135}$$

Conclusion: we find again that the propagation amplitude out the light-cone is exponentially decaying but nonzero!!! However, instead of asking whether particles can propagate over space-like intervals, we should ask whether a measurement performed at one point can affect a measurement at another point space-like separated. Measuring the field $\phi(x)$ by evaluating the commutator $[\phi(x), \phi(y)]$: if it vanishes, one measurement cannot affect the other. Moreover, if $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$, causality is generally preserved, since commutators involving any function of $\phi(x)$, e.g. $\Pi(x) = \partial_t \phi(x)$, will also vanish.

Consider the commutator:

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}[(a_{\vec{p}}e^{-ip.x} + a^\dagger_{\vec{p}}e^{ip.x}), (a_{\vec{q}}e^{-iq.y} + a^\dagger_{\vec{q}}e^{iq.y})] \tag{136}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(e^{-ip.(x-y)} - e^{ip.(x-y)}) = D(x-y) - D(y-x). \tag{137}$$

If we consider a space-like separation, $(x-y)^2 < 0$, we can perform a Lorentz transformation on the second term, $(x-y) \to -(x-y)$. Therefore, the commutator vanishes, and causality is preserved! On the other hand, if $(x-y)^2 > 0$, there is no Lorentz transformation that takes $(x-y) \to -(x-y)$. In this scenario and for $\vec{x} - \vec{y} = 0$, the amplitude is nonzero ($\sim e^{-imt} - e^{imt}$). This means that no measurement in the KG theory can affect another one outside the light-cone.

References

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[2] David Tong, *Lectures on QFT*, webpage