Quantum theory of scattering by a central potential

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1 Notation and Conventions

Throughout the text, d denotes the number of spatial dimensions. The fundamental parameter is

$$\nu \equiv \frac{d-2}{2}\,, \tag{1}$$

so that $\nu = \frac{1}{2}$ for $d = 3$. A closely related quantity, the "effective angular momentum", is

$$\lambda \equiv \ell + \nu = \ell + \frac{d-2}{2}\,. \tag{2}$$

The surface area of the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{3}$$

To derive this formula, we observe that the Gaussian integral in d dimensions factorises into a radial integral times the solid angle:

$$\pi^{d/2} = \int_{\mathbb{R}^d} \mathrm{e}^{-r^2}\mathrm{d}^d r = \Omega_d \int_0^\infty r^{d-1}\mathrm{e}^{-r^2}\mathrm{d}r = \frac{\Omega_d}{2}\,\Gamma(d/2), \tag{4}$$

using $t = r^2$ in the radial integral. Isolating Ω_d yields (3).

We single out $\theta \equiv \theta_1 \in [0,\pi]$ as the **scattering angle** (the angle between $\hat{\mathbf{r}}$ and the beam axis $\hat{\mathbf{e}}_d$). For any axially symmetric function $g(\theta)$, the full angular integral reduces to

$$\int_{S^{d-1}} g(\theta)\,\mathrm{d}\Omega_{d-1} = \omega_{d-1}\int_0^\pi g(\theta)\sin^{d-2}\theta\,\mathrm{d}\theta = \omega_{d-1}\int_{-1}^1 g(x)(1-x^2)^{\nu-1/2}\mathrm{d}x, \tag{5}$$

where $x = \cos\theta$ and the transverse solid angle is $\omega_{d-1} = \Omega_{d-1} = 2\pi^{(d-1)/2}/\Gamma((d{-}1)/2)$.

Part I
The Universal Framework

The whole content of this part is independent of the choice of potential $V(r)$. We construct the framework once; in Part II we apply it to specific potentials.

2 Conditions on the Potential

Before assembling the framework, we explicitly state the conditions that $V(r)$ must satisfy. The principal one is the **short-range condition**, which underpins the entire construction.

2.1 The short-range condition

A central potential $V(r)$ is **short-range** if

$$\lim_{r\to\infty} r\,V(r) = 0\,. \tag{6}$$

Equivalently, $V(r)$ decays faster than $1/r$ at infinity: $V(r) = o(1/r)$ as $r \to \infty$. Examples: $V \propto \mathrm{e}^{-\mu r}/r$ (Yukawa), $V \propto 1/r^n$ with $n \geq 2$, any compactly supported V, any Gaussian. The counter-example is the Coulomb potential itself, $V \propto 1/r$, for which $rV(r) = \text{const} \neq 0$.

2.2 Why the short-range condition is essential

The condition enters at three crucial points of the framework. First, in the **asymptotic boundary condition**:

$$\psi(\mathbf{r}) \xrightarrow{r\to\infty} \mathrm{e}^{\mathrm{i}kz} + f(\theta)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r^{(d-1)/2}}\,, \tag{7}$$

which presupposes that, at large r, the particle is effectively free — moving as a plane wave plus outgoing spherical waves. This requires V to become negligible at large distances sufficiently fast not to distort the asymptotic phase.

Second, in the **existence of phase shifts via matching**: the phase shifts δ_ℓ are defined by matching the interior solution (with V) with the free solutions J_λ, N_λ in an exterior region where $V \approx 0$. This matching only makes sense if there is some R_0 beyond which V is negligible.

Third, in the **convergence of the partial-wave sum**: the expansion $f(\theta) = \sum_\ell (\ldots)\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell\, C_\ell^\nu(\cos\theta)$ converges pointwise because the threshold behaviour $\delta_\ell \sim k^{2\ell+d-2}$ suppresses the contributions of large ℓ. This threshold behaviour depends on the short-range condition.

The three points above fail for the Coulomb potential: the asymptotic wave function acquires a logarithmic phase $\mathrm{e}^{\mathrm{i}\eta\ln(2kr)}$ that never disappears; the matching with free Bessel functions fails; and the partial-wave sum diverges. Coulomb scattering is a qualitatively distinct problem, treated in Section 12.

2.3 The critical line: why exactly $1/r$?

The power $1/r$ is the limiting case — the exact boundary between short- and long-range behaviour. To see this, consider $V(r) \sim C/r^n$ at large r. The phase accumulated by the wave function between R and ∞ is

$$\Delta\phi \sim \int_R^\infty \left[\sqrt{k^2 - U(r)} - k\right]\mathrm{d}r \approx -\frac{1}{2k}\int_R^\infty U(r)\,\mathrm{d}r \propto \int_R^\infty \frac{1}{r^n}\mathrm{d}r\,. \tag{8}$$

This integral converges for $n > 1$ (short range: a finite phase correction) and diverges for $n \le 1$ (long range: an infinite phase at infinity). The case $n = 1$ diverges logarithmically, which is exactly the logarithmic phase $\eta\ln(2kr)$ found in the Coulomb case.

2.4 Other conditions and summary

For completeness, we also require: (i) that $V(r)$ be real, which guarantees $\hat{H}$ Hermitian and the S-matrix unitary; (ii) that it be regular at the origin ($r^2V(r) \to 0$), guaranteeing that the centrifugal barrier $(\lambda^2 - \frac{1}{4})/r^2$ dominates near the origin; and (iii) that it be locally integrable, $\int_0^{R_0} r^{d-1}|V(r)|\mathrm{d}r < \infty$.

In summary, throughout the document, $V(r)$ is assumed real, spherically symmetric (central), locally integrable, regular at the origin, and above all **short-range**: $rV(r) \to 0$ as $r \to \infty$. The short-range condition is the most restrictive and is the one whose failure (for the Coulomb potential) forces a separate treatment.

3 Hyperspherical Coordinates and the Laplacian

3.1 Definition of the coordinate system

A point $\mathbf{r} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is parametrised by a radial coordinate $r \geq 0$ and $d-1$ angles $(\theta_1, \theta_2, \ldots, \theta_{d-2}, \varphi)$ via

$$\begin{aligned} x_1 &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-2} \cos\varphi, \\ x_2 &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-2} \sin\varphi, \\ x_k &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-k} \cos\theta_{d-k+1}, \quad 3 \leq k \leq d-1, \\ x_d &= r \cos\theta_1, \end{aligned} \tag{9}$$

with $r \geq 0$, $\theta_j \in [0, \pi]$ for $j = 1, \ldots, d-2$, and $\varphi \in [0, 2\pi)$. The Jacobian of the transformation (9) is

$$J = r^{d-1} \prod_{j=1}^{d-2} \sin^{d-1-j}\theta_j, \tag{10}$$

verifiable by induction in d. The volume element is therefore

$$\mathrm{d}^d r = r^{d-1} \mathrm{d}r \, \mathrm{d}\Omega_d, \qquad \mathrm{d}\Omega_d = \prod_{j=1}^{d-2} \sin^{d-1-j}\theta_j \, \mathrm{d}\theta_j \, \mathrm{d}\varphi. \tag{11}$$

3.2 The Laplacian in hyperspherical coordinates

In any orthogonal curvilinear coordinate system, the Laplacian splits into a radial part and an angular part. In d-dimensional hyperspherical coordinates,

$$\nabla_d^2 = \frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left(r^{d-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \hat{\Delta}_d \,, \tag{12}$$

where $\hat{\Delta}_d$ is a purely angular operator on S^{d-1} (the Laplace–Beltrami operator on the sphere). The radial part can be rewritten as

$$\frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left(r^{d-1} \frac{\partial}{\partial r} \right) = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}. \tag{13}$$

3.3 The angular operator $\hat{\Delta}_d$

The operator $\hat{\Delta}_d$ acts on functions on S^{d-1} and can be defined recursively. Writing the set of d-dimensional angles as (θ_1, Ω'), where Ω' are the $d-2$ remaining angles parametrising S^{d-2}:

$$\hat{\Delta}_d = \frac{1}{\sin^{d-2}\theta_1} \frac{\partial}{\partial \theta_1}\left(\sin^{d-2}\theta_1 \frac{\partial}{\partial \theta_1} \right) + \frac{1}{\sin^2\theta_1} \hat{\Delta}_{d-1}, \tag{14}$$

with the base case $\hat{\Lambda}_2^2 = \partial^2/\partial\varphi^2$. For $d = 3$ this reproduces the familiar form

$$\hat{\Lambda}_3^2 = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \,, \tag{15}$$

consistent with $\hat{\mathbf{L}}^2 = -\hbar^2 \hat{\Lambda}_3^2$.

4 Separation into Radial and Angular Parts

We now return to the time-independent Schrödinger equation for a central potential $V(\mathbf{r}) = V(r)$ and show how the radial and angular equations both emerge from a single separation of variables.

4.1 The Schrödinger equation in hyperspherical coordinates

A particle of mass m in a central potential $V(r)$, with energy $E > 0$, satisfies

$$\left[-\frac{\hbar^2}{2m}\nabla_d^2 + V(r)\right]\psi(\mathbf{r}) = E\,\psi(\mathbf{r}). \tag{16}$$

Defining $k = \sqrt{2mE}/\hbar$ and $U(r) = 2mV(r)/\hbar^2$, we can rewrite it as $(\nabla_d^2 + k^2 - U)\psi = 0$. Using the decomposition (12):

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\,\hat{\Delta}_d\,\psi\right] + V(r)\,\psi = E\,\psi. \tag{17}$$

Everything in this equation depends on both r and the angular coordinates $\Omega = (\theta_1, \ldots, \theta_{d-2}, \varphi)$. The key observation is that $V(r)$ depends only on r, while $\hat{\Delta}_d$ acts only on Ω.

4.2 Spherical symmetry motivates the separation

Before any algebra, let us see why the radial and angular parts separate. A central potential $V(r)$ has a crucial geometric property: its level sets are concentric spheres (Figure 1). Nothing in the problem distinguishes different directions — the potential has the same value at every point of a given sphere.

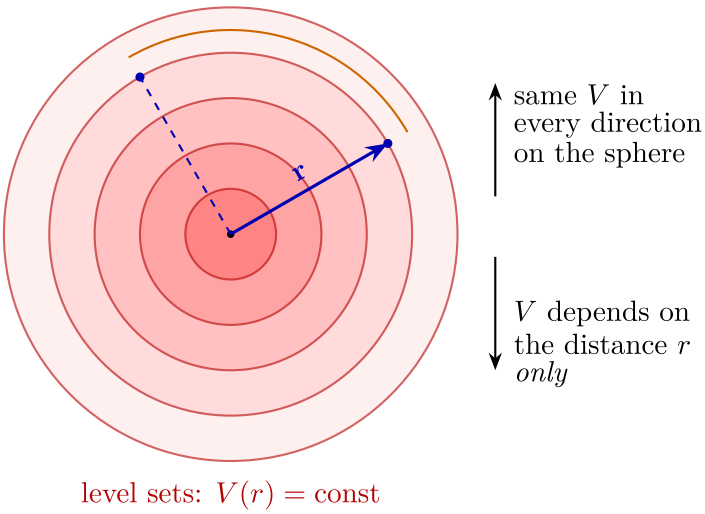

Figure 1: Spherical symmetry of a central potential. The level sets $V(r) = \text{const}$ are concentric spheres. On any given sphere, the potential takes the same value at all points. The problem has two independent degrees of freedom: motion along r and motion on the sphere. These decouple, which is why $\psi(\mathbf{r}) = R(r)\,Y(\hat{\mathbf{r}})$ is the natural ansatz.

The consequence: the wave function separates as a product $\psi(\mathbf{r}) = R(r)\,Y(\hat{\mathbf{r}})$, where $R(r)$ describes how the amplitude varies with distance (it senses the potential $V(r)$) and

$Y(\hat{\mathbf{r}})$ describes the angular pattern (sensing only the geometry of the sphere — hence universal).

4.3 The separation ansatz $\psi = R(r)\,Y(\Omega)$

Substituting $\psi(r,\Omega) = R(r)\,Y(\Omega)$ into (17):

$$-\frac{\hbar^2}{2m}\left[\frac{Y}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{R}{r^2}\,\hat{\Delta}_d\,Y\right] + V(r)\,R\,Y = E\,R\,Y. \tag{18}$$

Dividing by $R\,Y$ and multiplying by $-2mr^2/\hbar^2$:

$$\underbrace{\frac{r^2}{R}\,\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{2m}{\hbar^2}\big[V(r) - E\big]\,r^2}_{\text{depends only on } r} = -\underbrace{\frac{1}{Y}\,\hat{\Delta}_d\,Y}_{\text{depends only on } \Omega}. \tag{19}$$

The left-hand side is a function of r alone; the right-hand side, of Ω alone. Since they are equal for all r and all Ω, both must equal a common constant, which we denote λ.

4.4 The angular eigenvalue equation

Equating the right-hand side of (19) to λ:

$$\hat{\Delta}_d\,Y(\Omega) = -\lambda\,Y(\Omega). \tag{20}$$

This is an eigenvalue equation for the Laplace–Beltrami operator on S^{d-1}: the angular part of the wave function must be an eigenfunction of $\hat{\Delta}_d$ with eigenvalue $-\lambda$. At this stage, λ is an undetermined separation constant; its admissible values will be fixed by the requirement that Y be single-valued and normalisable on S^{d-1}.

4.5 The radial equation

Equating the left-hand side of (19) to λ, dividing by r^2 and restoring the original form:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{\lambda}{r^2}\,R\right] + V(r)\,R = E\,R. \tag{21}$$

This is the radial Schrödinger equation in d dimensions. The angular separation constant λ enters as a centrifugal-like term. Once the angular equation is solved and $\lambda = \ell(\ell+d-2)$ is determined (which we shall do next), (21) becomes a closed ODE for $R(r)$.

The angular equation (20) is **universal**: it is the same for every central potential, every energy, and even for bound states vs. scattering states — it is purely a property of the geometry of the $(d-1)$-sphere. The radial equation (21), by contrast, is where all the specific physics of a potential enters. We solve the angular equation once (next) and then the radial equation separately for each $V(r)$ (Part II).

5 Hyperspherical Harmonics: Complete Derivation

Our task is to find all eigenvalues λ and eigenfunctions $Y(\Omega)$ of the angular equation (20). We proceed by successive separations of variables, exploiting the recursive structure (14) of $\hat{\Delta}_d$.

5.1 The separation chain and the azimuthal equation

We carry out the first separation in full detail, identify the pattern, and follow the chain down to the azimuthal angle φ.

First separation: isolating θ_1. We seek eigenfunctions of $\hat{\Delta}_d$ on S^{d-1}. Using (14), the equation $\hat{\Delta}_d Y = -\lambda Y$ becomes explicitly

$$\frac{1}{\sin^{d-2}\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^{d-2}\theta_1\frac{\partial Y}{\partial\theta_1}\right)+\frac{1}{\sin^2\theta_1}\hat{\Delta}_{d-1}Y=-\lambda Y, \tag{22}$$

where $\hat{\Delta}_{d-1}$ acts on the remaining $d-2$ angles $(\theta_2,\ldots,\theta_{d-2},\varphi)$. We try the product ansatz

$$Y(\theta_1,\theta_2,\ldots,\varphi)=\Theta_1(\theta_1)\;Z(\theta_2,\ldots,\theta_{d-2},\varphi). \tag{23}$$

Substituting, dividing by $\Theta_1 Z$ and multiplying by $\sin^2\theta_1$:

$$\underbrace{\frac{\sin^2\theta_1}{\Theta_1}\,\frac{1}{\sin^{d-2}\theta_1}\frac{\mathrm{d}}{\mathrm{d}\theta_1}\big(\sin^{d-2}\theta_1\,\Theta_1'\big)+\lambda\sin^2\theta_1}_{\text{depends only on }\theta_1}=-\underbrace{\frac{1}{Z}\hat{\Delta}_{d-1}Z}_{\text{depends only on }(\theta_2,\ldots,\varphi)}. \tag{24}$$

Both sides must equal a separation constant, $\lambda^{(d-1)}$. The right-hand side gives

$$\hat{\Delta}_{d-1}Z=-\lambda^{(d-1)}Z, \tag{25}$$

which has the same structure as $\hat{\Delta}_d Y=-\lambda Y$, but in one fewer dimension.

The pattern: a chain of eigenvalue equations. Applying the same procedure successively, we generate a chain of factorisations:

$$\begin{aligned}
Y^{(d)}(\theta_1,\theta_2,\ldots,\theta_{d-2},\varphi)&=\Theta_1(\theta_1)\;\cdot\;Y^{(d-1)}(\theta_2,\ldots,\varphi),\\
Y^{(d-1)}(\theta_2,\ldots,\varphi)&=\Theta_2(\theta_2)\;\cdot\;Y^{(d-2)}(\theta_3,\ldots,\varphi),\\
&\;\;\vdots\\
Y^{(3)}(\theta_{d-2},\varphi)&=\Theta_{d-2}(\theta_{d-2})\;\cdot\;\Phi(\varphi),
\end{aligned} \tag{26}$$

and the complete eigenfunction is the product

$$Y^{(d)}=\Theta_1(\theta_1)\,\Theta_2(\theta_2)\;\cdots\;\Theta_{d-2}(\theta_{d-2})\,\Phi(\varphi). \tag{27}$$

The last separation: the ODE for $\Phi(\varphi)$. After $d-3$ separations of this kind, we are left with the equation for $\hat{\Lambda}_3^2$ on S^2. A final separation $Y^{(3)}=\Theta_{d-2}(\theta_{d-2})\,\Phi(\varphi)$ produces, for the φ part:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=-\lambda_\varphi\,\Phi. \tag{28}$$

The equation for $\Phi(\varphi)$ is not assumed *a priori*: it emerges as the final separation constant in the chain that began with the full equation (22).

General solution. Try $\Phi(\varphi)=\mathrm{e}^{\mathrm{i}\alpha\varphi}$. Substituting: $(\mathrm{i}\alpha)^2=-\alpha^2=-\lambda_\varphi$, hence $\lambda_\varphi=\alpha^2$ and $\Phi(\varphi)=A\,\mathrm{e}^{\mathrm{i}\alpha\varphi}+B\,\mathrm{e}^{-\mathrm{i}\alpha\varphi}$.

Single-valuedness condition (periodicity). The wave function must be single-valued: $\Phi(\varphi + 2\pi) = \Phi(\varphi)$. Applied to $e^{i\alpha\varphi}$:

$$e^{2\pi i\alpha} = 1 \quad \Longrightarrow \quad \alpha \in \mathbb{Z}. \tag{29}$$

Taking $\alpha = m$ with $m = 0, \pm1, \pm2, \ldots$, we organise the solutions in the normalised basis

$$\Phi_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}, \qquad m \in \mathbb{Z}, \qquad \lambda_\varphi = m^2. \tag{30}$$

The normalisation $\int_0^{2\pi} |\Phi_m|^2 \, d\varphi = 1$ is immediate, and the orthogonality $\int_0^{2\pi} \Phi_{m'}^* \Phi_m \, d\varphi = \delta_{mm'}$ follows from $e^{i(m-m')2\pi} = 1$ for $m - m' \in \mathbb{Z}$.

This result holds in every dimension $d \geq 2$: the azimuthal angle always carries the quantum number $m \in \mathbb{Z}$, and its separation constant is always m^2. The value m^2 re-enters the equation for θ_{d-2} as a centrifugal term $m^2/\sin^2\theta_{d-2}$, whose solution produces a new constant that feeds the equation for θ_{d-3}, and so on down to θ_1.

5.2 The polar-angle equations: the general step

Each polar-angle equation has the same mathematical structure. Rather than repeating it $d - 3$ times, we formalise the general step by induction.

Suppose the eigenvalue problem on S^{d-2} (the sphere parametrised by $\Omega' = (\theta_2, \ldots, \theta_{d-2}, \varphi)$) has already been solved — with the azimuthal equation (30) as the ultimate base case:

$$\hat{\Delta}_{d-1} \, Y_{\ell'}^{(d-1)}(\Omega') = -\ell'(\ell' + d - 3) \, Y_{\ell'}^{(d-1)}(\Omega'), \qquad \ell' = 0, 1, 2, \ldots \tag{31}$$

We seek solutions of $\hat{\Delta}_d \, Y = -\lambda \, Y$ in the separated form

$$Y(\theta_1, \Omega') = \Theta(\theta_1) \; Y_{\ell'}^{(d-1)}(\Omega'). \tag{32}$$

Substituting via (14) and dividing by $\Theta \, Y_{\ell'}^{(d-1)}$:

$$\frac{1}{\sin^{d-2}\theta_1} \frac{d}{d\theta_1}\left(\sin^{d-2}\theta_1 \frac{d\Theta}{d\theta_1}\right) - \frac{\ell'(\ell' + d - 3)}{\sin^2\theta_1} \, \Theta + \lambda \, \Theta = 0. \tag{33}$$

This is the master ODE for each angular level. It remains to solve it.

5.3 Change of variable: $u = \cos\theta_1$

Take $u = \cos\theta_1 \in [-1, 1]$, with $\sin\theta_1 = \sqrt{1 - u^2}$ and $d/d\theta_1 = -\sin\theta_1 \, d/du$. We transform each piece of (33).

For the differential operator, write $s \equiv \sin\theta_1 = \sqrt{1 - u^2}$. The inner derivative becomes

$$s^{d-2} \frac{d\Theta}{d\theta_1} = s^{d-2} \cdot (-s) \, \Theta'(u) = -(1 - u^2)^{(d-1)/2} \, \Theta'(u). \tag{34}$$

Applying the outer derivative and dividing by s^{d-2} gives, after simplification:

$$\frac{1}{s^{d-2}} \frac{d}{d\theta_1}\left(s^{d-2} \frac{d\Theta}{d\theta_1}\right) = (1 - u^2) \, \Theta'' - (d - 1) \, u \, \Theta'. \tag{35}$$

The centrifugal term becomes

$$-\frac{\ell'(\ell' + d - 3)}{\sin^2\theta_1} \, \Theta = -\frac{\ell'(\ell' + d - 3)}{1 - u^2} \, \Theta. \tag{36}$$

Combining (35), (36) and the $+\lambda \, \Theta$ term, equation (33) becomes

$$(1 - u^2) \, \Theta'' - (d - 1) \, u \, \Theta' + \left[\lambda - \frac{\ell'(\ell' + d - 3)}{1 - u^2}\right] \Theta = 0. \tag{37}$$

5.4 Extracting the singular behaviour

Equation (37) has regular singular points at $u = \pm 1$ (the poles of the sphere). The term $\ell'(\ell' + d - 3)/(1 - u^2)$ diverges there, so we extract the expected singular behaviour by writing

$$\Theta(u) = (1 - u^2)^{\ell'/2} \, P(u). \tag{38}$$

This is the d-dimensional generalisation of the standard 3D trick, in which the solution of the associated Legendre equation is written as $(1 - u^2)^{|m|/2}$ times a polynomial.

Substituting (38) into (37) and working through each term. Writing $\sigma \equiv \ell'$ for compactness and $f(u) = (1 - u^2)^{\sigma/2}$:

$$f' = -\sigma \, u \, (1 - u^2)^{\sigma/2 - 1}, \tag{39}$$

$$\Theta' = (1 - u^2)^{\sigma/2 - 1} \big[-\sigma \, u \, P + (1 - u^2) P' \big], \tag{40}$$

$$\Theta'' = f'' \, P + 2 f' \, P' + f \, P''. \tag{41}$$

With $f'' = (1 - u^2)^{\sigma/2 - 2} \, \sigma [(\sigma - 1) u^2 - 1]$ one gets

$$\begin{aligned} \Theta'' = {} & (1 - u^2)^{\sigma/2 - 2} \, \sigma [(\sigma - 1) u^2 - 1] \, P \\ & + 2(-\sigma \, u)(1 - u^2)^{\sigma/2 - 1} \, P' + (1 - u^2)^{\sigma/2} \, P''. \end{aligned} \tag{42}$$

Substituting into (37) and dividing the entire equation by $(1 - u^2)^{\sigma/2}$, we collect the coefficients of P'', P' and P. The coefficient of P'' is simply $(1 - u^2)$. That of P' is

$$-2\sigma \, u - (d - 1) \, u = -(2\sigma + d - 1) \, u. \tag{43}$$

The coefficient of P requires care. Collecting the various contributions in $1/(1 - u^2)$ and the λ term:

$$\begin{aligned} & \frac{\sigma[(\sigma - 1) u^2 - 1]}{1 - u^2} + \frac{(d - 1)\sigma u^2}{1 - u^2} + \lambda - \frac{\sigma(\sigma + d - 3)}{1 - u^2} \\ & = \frac{\sigma}{1 - u^2} \Big\{ (\sigma + d - 2) u^2 - (\sigma + d - 2) \Big\} + \lambda \\ & = -\sigma(\sigma + d - 2) + \lambda. \end{aligned} \tag{44}$$

The key point: *the singular terms in* $1/(1 - u^2)$ *cancel exactly*, leaving a constant. This is precisely why the ansatz (38) works.

The resulting ODE for $P(u)$ is

$$(1 - u^2) \, P'' - (2\sigma + d - 1) \, u \, P' + \big[\lambda - \sigma(\sigma + d - 2) \big] \, P = 0, \tag{45}$$

with $\sigma = \ell'$.

5.5 Identification as the Gegenbauer equation and quantisation

The standard Gegenbauer (ultraspherical) equation with parameter $\alpha > 0$ and degree n is

$$(1 - u^2) \, P'' - (2\alpha + 1) \, u \, P' + n(n + 2\alpha) \, P = 0. \tag{46}$$

Comparing (45) with (46):

$$2\alpha + 1 = 2\sigma + d - 1 \quad \Longrightarrow \quad \alpha = \sigma + \frac{d - 2}{2} = \ell' + \frac{d - 2}{2}, \tag{47}$$

$$n(n + 2\alpha) = \lambda - \sigma(\sigma + d - 2). \tag{48}$$

Equation (46) has polynomial solutions — the **Gegenbauer polynomials** $C_n^{(\alpha)}(u)$ — if and only if n is a non-negative integer. Otherwise, the series solution diverges at $u = \pm 1$, producing functions that are not normalisable on S^{d-1}.

We therefore require $n \in \{0, 1, 2, \ldots\}$. Define $\ell \equiv n + \sigma = n + \ell'$, that is,

$$n = \ell - \ell', \qquad \ell \geq \ell' \geq 0. \tag{49}$$

Solving (48) for λ:

$$\begin{aligned}\lambda &= n(n + 2\alpha) + \sigma(\sigma + d - 2) \\ &= (\ell - \ell')(\ell + \ell' + d - 2) + \ell'(\ell' + d - 2) \\ &= \ell^2 + \ell(d - 2) = \ell(\ell + d - 2). \end{aligned} \tag{50}$$

We conclude:

$$\hat{\Delta}_d \, Y_\ell^{(d)}(\Omega) = -\ell(\ell + d - 2)\, Y_\ell^{(d)}(\Omega), \qquad \ell = 0, 1, 2, \ldots \tag{51}$$

The eigenvalue $\lambda = \ell(\ell + d - 2)$ depends only on ℓ and d, and is independent of ℓ' and the other lower quantum numbers. This is the counterpart of the familiar fact in $d = 3$ that the eigenvalue $\ell(\ell + 1)$ of $\hat{\mathbf{L}}^2$ does not depend on m.

For $d = 3$, $\lambda = \ell(\ell + 1)$. For $d = 2$, $\lambda = \ell(\ell + 0) = \ell^2 = m^2$, in agreement with the base case.

5.6 The complete angular eigenfunctions

Putting all the pieces together, the normalised solution of the θ_1 equation is

$$\Theta_{\ell,\ell'}(\theta_1) = \mathcal{N}_{\ell,\ell'}^{(d)} \; \sin^{\ell'}\theta_1 \; C_{\ell-\ell'}^{\ell'+(d-2)/2}(\cos\theta_1), \tag{52}$$

where $C_n^{(\alpha)}$ is the Gegenbauer polynomial and $\mathcal{N}_{\ell,\ell'}^{(d)}$ is the normalisation constant fixed by $\int_0^\pi |\Theta_{\ell,\ell'}|^2 \sin^{d-2}\theta_1 \, \mathrm{d}\theta_1 = 1$.

The **full hyperspherical harmonic** is then constructed recursively as

$$Y_{\ell_1\ell_2\cdots\ell_{d-2}m}^{(d)}(\theta_1, \ldots, \theta_{d-2}, \varphi) = \prod_{j=1}^{d-2} \Theta_{\ell_j,\ell_{j+1}}^{(j)}(\theta_j) \;\cdot\; \frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}, \tag{53}$$

with $\ell_{d-1} \equiv |m|$ and each factor $\Theta^{(j)}$ in the form (52) with the dimension and quantum numbers appropriate to the j-th level of the recursion.

For $d = 3$ the quantum numbers are $(\ell_1, m) = (\ell, m)$, and the single Θ factor is $\Theta_{\ell,|m|}(\theta) \propto \sin^{|m|}\theta \, C_{\ell-|m|}^{|m|+1/2}(\cos\theta)$. The Gegenbauer polynomial $C_n^{|m|+1/2}(\cos\theta)$ is proportional to the associated Legendre function $P_\ell^{|m|}(\cos\theta)$, recovering the standard spherical harmonic.

5.7 Quantum numbers and degeneracy

A complete set of quantum numbers labelling a hyperspherical harmonic on S^{d-1} is $(\ell_1, \ell_2, \ldots, \ell_{d-2}, m)$, subject to the nesting condition

$$\ell \equiv \ell_1 \geq \ell_2 \geq \cdots \geq \ell_{d-2} \geq |m| \geq 0, \qquad m \in \mathbb{Z}. \tag{54}$$

The degeneracy $N(\ell, d)$ — the number of linearly independent harmonics for a given ℓ on S^{d-1} — is obtained by counting these sequences. The result, derived by induction in d, is

$$N(\ell, d) = \frac{(2\ell + d - 2)(\ell + d - 3)!}{\ell!\,(d-2)!}, \qquad \ell \geq 0. \tag{55}$$

For $d = 3$: $N(\ell, 3) = 2\ell + 1$ (the familiar set of spherical harmonics Y_ℓ^m with $m = -\ell, \ldots, +\ell$). For $d = 4$: $N(\ell, 4) = (\ell + 1)^2$, the well-known degeneracy on S^3.

6 The Radial Equation: Further Development

6.1 The radial equation with the angular eigenvalue inserted

Having solved the angular problem, we substitute $\lambda = \ell(\ell + d - 2)$ into (21). Expanding the radial derivative via (13):

$$-\frac{\hbar^2}{2m}\left[\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{d-1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell + d - 2)}{r^2}R\right] + V(r)\,R = E\,R. \tag{56}$$

6.2 Reduction to an effective one-dimensional problem

It is convenient to eliminate the first-derivative term by substituting

$$R(r) = \frac{u(r)}{r^{(d-1)/2}}. \tag{57}$$

One verifies by direct differentiation that, if R satisfies (56), then $u(r)$ satisfies

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\mathcal{L}(\mathcal{L}+1)}{r^2}\right]u = E\,u, \tag{58}$$

with the *effective angular-momentum parameter*

$$\mathcal{L} \equiv \ell + \frac{d-3}{2}. \tag{59}$$

For the derivation, substituting $R = u/r^{(d-1)/2}$:

$$R' = \frac{u'}{r^{(d-1)/2}} - \frac{d-1}{2}\frac{u}{r^{(d+1)/2}}, \tag{60}$$

$$R'' = \frac{u''}{r^{(d-1)/2}} - \frac{d-1}{r}\frac{u'}{r^{(d-1)/2}} + \frac{(d-1)(d+1)}{4}\frac{u}{r^{(d+3)/2}}. \tag{61}$$

Inserting into (56) and simplifying:

$$R'' + \frac{d-1}{r}R' = \frac{u''}{r^{(d-1)/2}} - \frac{(d-1)(d-3)}{4}\frac{u}{r^{(d+3)/2}}. \tag{62}$$

The centrifugal term in (56) contributes $\ell(\ell + d - 2)\,u/r^{(d+3)/2}$. Combining:

$$\frac{\ell(\ell + d - 2) + \frac{(d-1)(d-3)}{4}}{r^2} = \frac{\left(\ell + \frac{d-3}{2}\right)\left(\ell + \frac{d-1}{2}\right)}{r^2} = \frac{\mathcal{L}(\mathcal{L}+1)}{r^2}, \tag{63}$$

with $\mathcal{L} = \ell + (d-3)/2$, confirming (58) and (59).

Equation (58) has *exactly* the form of the one-dimensional radial Schrödinger equation in three dimensions, but with the angular-momentum quantum number ℓ replaced by $\mathcal{L} = \ell + (d-3)/2$. This is the key structural observation: the central-potential problem in d dimensions maps onto a family of effective 1D problems labelled by $\mathcal{L}$.

6.3 Normalisation condition and effective potential

Since $\mathrm{d}^d r = r^{d-1}\mathrm{d}r\,\mathrm{d}\Omega_d$ and the hyperspherical harmonics are orthonormal on S^{d-1}, the normalisation of the full wave function reduces to $\int_0^\infty |u(r)|^2\,\mathrm{d}r = 1$, and $u(r)$ plays the role of a standard wave function in $L^2(0,\infty)$. The boundary conditions are $u(0) = 0$ (regularity at the origin, since $R = u/r^{(d-1)/2}$ must be finite) and $u(r) \to 0$ as $r \to \infty$ (for bound states).

For any central potential, the reduced equation (58) involves the effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{\hbar^2}{2m}\frac{\mathcal{L}(\mathcal{L}+1)}{r^2}. \tag{64}$$

The centrifugal barrier $\propto 1/r^2$ grows both with ℓ and with d. In the scattering problem, it dominates near the origin (except in $d = 2$ with $\ell = 0$).

6.4 Reduction to the Bessel equation

Setting $V = 0$ in (58) and using $\rho = kr$, with the notation $\lambda = \ell + \nu = \mathcal{L} + \frac{1}{2}$ (caution: λ here is the order of the Bessel function, distinct from the angular separation constant used in §4):

$$\frac{\mathrm{d}^2 u_\ell}{\mathrm{d}\rho^2} + \left[1 - \frac{\lambda^2 - \frac{1}{4}}{\rho^2}\right] u_\ell = 0. \tag{65}$$

The substitution $u_\ell = \sqrt{\rho}\,w$ converts this equation into the **Bessel equation of order** λ:

$$\rho^2 w'' + \rho w' + (\rho^2 - \lambda^2) w = 0. \tag{66}$$

Equation (66) is a second-order ODE, and therefore admits two linearly independent solutions. These are:

- $J_\lambda(\rho)$, the **Bessel function of the first kind**, defined by the series

$$J_\lambda(\rho) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\lambda+1)} \left(\frac{\rho}{2}\right)^{2n+\lambda}. \tag{67}$$

  It is regular at the origin, with behaviour $J_\lambda(\rho) \sim \rho^\lambda/[2^\lambda \Gamma(\lambda+1)]$ as $\rho \to 0$, and oscillates as $\sim \sqrt{2/(\pi\rho)}$ at large ρ.

- $N_\lambda(\rho)$, the **Bessel function of the second kind**, or **Neumann function**, defined by the linear combination

$$N_\lambda(\rho) = \frac{\cos(\lambda\pi)\,J_\lambda(\rho) - J_{-\lambda}(\rho)}{\sin(\lambda\pi)} \tag{68}$$

  (with the definition extended by limit when λ is an integer). It is *singular* at the origin, diverging as $N_\lambda(\rho) \sim -2^\lambda \Gamma(\lambda)/(\pi\rho^\lambda)$ as $\rho \to 0$. It also oscillates at large ρ, but with phase shifted by $\pi/2$ relative to J_λ.

Remark 6.1 (Notation convention)**.** Part of the mathematical literature (Wikipedia, Wolfram, Abramowitz & Stegun) denotes the function of the second kind by $Y_\lambda(\rho)$. The convention in physics — adopted by Arfken, Jackson, Sakurai and most quantum-mechanics and scattering textbooks — is $N_\lambda(\rho)$, which we shall use throughout the text.

Returning to the variable $u_\ell = \sqrt{\rho}\, w$, the two linearly independent solutions of the free radial equation (65) are:

$$u_\ell^{(1)}(\rho) = \sqrt{\rho}\, J_\lambda(\rho) \quad \text{(regular at } \rho = 0\text{)}, \qquad u_\ell^{(2)}(\rho) = \sqrt{\rho}\, N_\lambda(\rho) \quad \text{(singular at } \rho = 0\text{)}. \tag{69}$$

6.5 Asymptotic behaviour

As $\rho \to \infty$, using $J_\lambda(\rho) \sim \sqrt{2/(\pi\rho)}\cos(\rho - \lambda\pi/2 - \pi/4)$:

$$\sqrt{\rho}\, J_\lambda(\rho) \xrightarrow{\rho\to\infty} \sqrt{\frac{2}{\pi}} \sin\left(\rho - \frac{\ell\pi}{2} - \frac{(d-3)\pi}{4}\right), \tag{70}$$

by the identity $\cos(\rho - \lambda\pi/2 - \pi/4) = \sin(\rho - \ell\pi/2 - (d-3)\pi/4)$ which follows from $\lambda\pi/2 + \pi/4 = \ell\pi/2 + (d-1)\pi/4$. Similarly, $\sqrt{\rho}\, N_\lambda \to -\sqrt{2/\pi}\cos(\rho - \ell\pi/2 - (d-3)\pi/4)$.

As $\rho \to 0$:

$$J_\lambda(\rho) \sim \frac{\rho^\lambda}{2^\lambda \Gamma(\lambda+1)}, \qquad N_\lambda(\rho) \sim -\frac{2^\lambda \Gamma(\lambda)}{\pi \rho^\lambda}. \tag{71}$$

For a free particle everywhere, N_λ is excluded by regularity at the origin. But in scattering, N_λ appears in the exterior region ($r > R_0$, where $V = 0$) because the origin is *inside* the potential region. The amount of N_λ mixed in encodes the phase shift.

For compactness, we define

$$\hat{j}_\ell^{(d)}(\rho) \equiv \frac{J_{\ell+\nu}(\rho)}{(\rho/2)^\nu}, \tag{72}$$

the natural generalisation of the spherical Bessel function in d dimensions. In $d = 3$ it reduces to $\sqrt{2/\pi}\, j_\ell(\rho)$.

7 The Rayleigh Expansion of the Plane Wave

We now derive a central result: the decomposition of a plane wave into partial waves.

7.1 Statement and strategy

The result to be established is

$$e^{ikr\cos\theta} = \Gamma(\nu) \sum_{\ell=0}^{\infty} (\ell+\nu)\, i^\ell\, \hat{j}_\ell^{(d)}(kr)\, C_\ell^\nu(\cos\theta). \tag{73}$$

By axial symmetry, $e^{i\rho\cos\theta}$ (with $\rho = kr$) depends only on r and θ, and so expands as $e^{i\rho\cos\theta} = \sum_\ell a_\ell(\rho)\, C_\ell^\nu(\cos\theta)$. Gegenbauer orthogonality gives the coefficients:

$$a_\ell(\rho) = \frac{1}{h_\ell^{(\nu)}} I_\ell(\rho), \qquad I_\ell(\rho) \equiv \int_{-1}^{1} e^{i\rho x}\, C_\ell^\nu(x)\, (1-x^2)^{\nu-1/2} dx, \tag{74}$$

where $h_\ell^{(\nu)} = \pi\, 2^{1-2\nu}\Gamma(\ell+2\nu)/[\ell!\,(\ell+\nu)\Gamma(\nu)^2]$. The whole derivation reduces to evaluating $I_\ell(\rho)$.

7.2 Evaluation of $I_\ell(\rho)$

Step 1: Apply Rodrigues' formula. The Rodrigues formula for the Gegenbauer polynomials allows us to write

$$I_\ell = \frac{(-1)^\ell}{2^\ell\,\ell!}\,\frac{\Gamma(\nu+\frac{1}{2})\,\Gamma(\ell+2\nu)}{\Gamma(2\nu)\,\Gamma(\ell+\nu+\frac{1}{2})}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\rho x}\,\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\big[(1-x^2)^{\ell+\nu-1/2}\big]\,\mathrm{d}x, \tag{75}$$

after cancellation of the weight $(1-x^2)^{\nu-1/2}$ against $(1-x^2)^{-(\nu-1/2)}$ from the formula.

Step 2: Integrate by parts ℓ times. The boundary terms vanish, since $(1-x^2)^{\ell+\nu-1/2}$ and its derivatives up to order $\ell-1$ vanish at $x=\pm1$. Each integration by parts produces a factor $-\mathrm{i}\rho$:

$$I_\ell = \frac{(\mathrm{i}\rho)^\ell}{2^\ell\,\ell!}\,\frac{\Gamma(\nu+\frac{1}{2})\,\Gamma(\ell+2\nu)}{\Gamma(2\nu)\,\Gamma(\ell+\nu+\frac{1}{2})}\,\mathcal{J}_\mu(\rho), \tag{76}$$

with $\mathcal{J}_\mu(\rho)=\int_{-1}^{1}\mathrm{e}^{\mathrm{i}\rho x}(1-x^2)^\mu\mathrm{d}x$ and $\mu=\ell+\nu-\frac{1}{2}$.

Step 3: Compute $\mathcal{J}_\mu(\rho)$ via series. Since $(1-x^2)^\mu$ is even, only the cosine part contributes. Expanding $\cos(\rho x)$ in series and using the Beta function for the integrals $\int_0^1 x^{2n}(1-x^2)^\mu\mathrm{d}x$, one identifies the Bessel-function series:

$$\mathcal{J}_\mu(\rho) = \frac{\sqrt{\pi}\,\Gamma(\mu+1)}{(\rho/2)^{\mu+1/2}}\,J_{\mu+1/2}(\rho). \tag{77}$$

Step 4: Substitute and simplify. With $\mu+\frac{1}{2}=\lambda=\ell+\nu$ and $\Gamma(\mu+1)=\Gamma(\ell+\nu+\frac{1}{2})$, the cancellations in the result of Step 2 give

$$I_\ell(\rho) = \mathrm{i}^\ell\,\frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\ell+2\nu)}{2^\nu\,\ell!\,\Gamma(2\nu)}\,\hat{\jmath}_\ell^{(d)}(\rho). \tag{78}$$

Step 5: Compute $a_\ell = I_\ell/h_\ell^{(\nu)}$. Applying the Legendre duplication formula $\Gamma(2\nu) = 2^{2\nu-1}\Gamma(\nu)\Gamma(\nu+\frac{1}{2})/\sqrt{\pi}$ and simplifying the Γ factors and powers of 2:

$$a_\ell(\rho) = \Gamma(\nu)\,(\ell+\nu)\,\mathrm{i}^\ell\,\hat{\jmath}_\ell^{(d)}(\rho), \tag{79}$$

which substituted into the expansion gives (73).

7.3 Physical interpretation

Each partial wave ℓ in the plane wave carries: an angular pattern $C_\ell^\nu(\cos\theta)$; a radial function $\hat{\jmath}_\ell^{(d)}(kr)$ — the regular free solution (regular because the plane wave is well-behaved at $r=0$); a weight $(\ell+\nu)$ — the amount of angular momentum ℓ present; and a kinematic phase i^ℓ. The absence of N_λ is crucial: when a potential introduces a mixture of N_λ in the exterior region, that mixture is due entirely to scattering.

8 The Scattering Framework

This section is universal: it applies to any short-range central potential. The specific potential enters only through the phase shifts δ_ℓ.

8.1 The scattering boundary condition

At large distances from the target, the scattering wave function must take the form

$$\psi(\mathbf{r}) \xrightarrow{r\to\infty} \mathrm{e}^{\mathrm{i}kz} + f(\theta)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r^{(d-1)/2}}\,, \tag{80}$$

where $z = r\cos\theta$ and $f(\theta)$ is the **scattering amplitude**. This condition is the starting point of the entire theory, and deserves justification at three levels: the physical meaning of each term, the origin of the exponent $1/r^{(d-1)/2}$ via probability conservation, and the formal justification via the Green's function.

Physical meaning of each term. We prepare a particle very far from the target, with a well-defined momentum $\mathbf{p} = \hbar k\,\hat{\mathbf{e}}_d$ along the beam axis. Before the interaction, it is described by a plane wave $\mathrm{e}^{\mathrm{i}kz}$ — a momentum eigenstate, propagating in the $+z$ direction. After interacting with $V(r)$ in the region $r \lesssim R_0$, an additional component arises: the *outgoing wave*, which propagates radially outward from the scattering centre. Since the potential is central and the beam has axial symmetry around the beam axis, the outgoing wave

- propagates radially: the factor $\mathrm{e}^{\mathrm{i}kr}$ is the spherical analogue of $\mathrm{e}^{\mathrm{i}kz}$, a wave of wavenumber k moving away from the origin;
- carries an angular distribution $f(\theta)$, the scattering amplitude, encoding how much flux exits in each direction;
- decays in amplitude as $1/r^{(d-1)/2}$ — the exponent needs to be justified, which we do next.

The condition (94) states, then: the full wave function, far from the target, is the superposition of the incident plane wave (which keeps passing through) plus the scattered spherical wave.

Origin of the exponent $1/r^{(d-1)/2}$: probability conservation. The power $1/r^{(d-1)/2}$ is not conventional: it is the only one compatible with flux conservation in d dimensions. Consider a generic spherical wave

$$\psi_{\mathrm{sc}}(\mathbf{r}) \sim A(\theta)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r^{\alpha}} \tag{81}$$

with exponent α to be determined. The probability current density is

$$\mathbf{j} = \frac{\hbar}{m}\,\mathrm{Im}[\psi^*\nabla\psi]. \tag{82}$$

Computing the radial component $j_r = \hat{\mathbf{r}}\cdot\mathbf{j}$ at large r (where the radial derivative of $\mathrm{e}^{\mathrm{i}kr}/r^{\alpha}$ is dominated by the ik term, since $\alpha/r \ll k$):

$$\partial_r\psi_{\mathrm{sc}} = A(\theta)\left(\frac{\mathrm{i}k}{r^{\alpha}} - \frac{\alpha}{r^{\alpha+1}}\right)\mathrm{e}^{\mathrm{i}kr} \xrightarrow{r\to\infty} \mathrm{i}k\,A(\theta)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r^{\alpha}}, \tag{83}$$

$$\psi^*_{\mathrm{sc}}\,\partial_r\psi_{\mathrm{sc}} \xrightarrow{r\to\infty} \mathrm{i}k\,\frac{|A(\theta)|^2}{r^{2\alpha}}, \tag{84}$$

$$j_r = \frac{\hbar}{m}\,\mathrm{Im}[\psi^*_{\mathrm{sc}}\partial_r\psi_{\mathrm{sc}}] = \frac{\hbar k}{m}\,\frac{|A(\theta)|^2}{r^{2\alpha}} = v\,\frac{|A(\theta)|^2}{r^{2\alpha}}. \tag{85}$$

The rate of particles crossing a sphere of radius r per unit time is the flux times the area element of the sphere in d dimensions. The area element is $dA = r^{d-1}\, d\Omega_{d-1}$ (it is the r^{d-1} that appears in $d^d r = r^{d-1}\, dr\, d\Omega_{d-1}$), and so

$$d\dot{N} = j_r\, dA = v\, |A(\theta)|^2\, \frac{r^{d-1}}{r^{2\alpha}}\, d\Omega_{d-1}. \tag{86}$$

For this rate to be *finite and independent of r* — required by probability conservation, since there can be no build-up nor loss of particles in the space between target and detector — the exponent in r must cancel:

$$r^{d-1-2\alpha} = \text{const} \quad \Longrightarrow \quad d-1-2\alpha = 0 \quad \Longrightarrow \quad \boxed{\alpha = \frac{d-1}{2}\,.} \tag{87}$$

Checking familiar cases: in $d = 3$, $\alpha = 1$, recovering the usual $\mathrm{e}^{\mathrm{i}kr}/r$. In $d = 2$, $\alpha = 1/2$, giving $\mathrm{e}^{\mathrm{i}kr}/\sqrt{r}$ — the cylindrical wave of planar diffraction problems. In $d = 4$, $\alpha = 3/2$.

Note that this derivation also explains *why* the exponent is the same in the boundary condition (94): identifying $A(\theta) \equiv f(\theta)$, we ensure that the number of particles scattered per unit time,

$$\dot{N}_{\mathrm{sc}} = v \int |f(\theta)|^2\, d\Omega_{d-1}, \tag{88}$$

is a finite quantity independent of the distance at which we place the detector — exactly what we need to define a physical cross section.

Formal justification: the Green's function. The two arguments above are physically convincing, but the condition (94) also emerges automatically from the mathematical structure of the problem. Rewriting the Schrödinger equation (16) as an inhomogeneous equation for the free part,

$$(\nabla_d^2 + k^2)\psi(\mathbf{r}) = U(r)\, \psi(\mathbf{r}), \tag{89}$$

with $U = 2mV/\hbar^2$ treated as a "source", the formal solution can be written as

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int G_+(\mathbf{r}, \mathbf{r}')\, U(r')\, \psi(\mathbf{r}')\, d^d r', \tag{90}$$

where $\psi_0 = \mathrm{e}^{\mathrm{i}kz}$ is the free incident wave (a solution of the homogeneous equation) and G_+ is the **retarded Green's function** of the Helmholtz operator $(\nabla_d^2 + k^2)$, defined by

$$(\nabla_d^2 + k^2)\, G_+(\mathbf{r}, \mathbf{r}') = \delta^{(d)}(\mathbf{r} - \mathbf{r}'), \tag{91}$$

with the condition that G_+ contains only *outgoing* waves (hence the + subscript).

In d dimensions, G_+ can be written in closed form using Hankel functions, and its asymptotic behaviour for $r \gg r'$ is

$$G_+(\mathbf{r}, \mathbf{r}') \xrightarrow{r\to\infty} -\frac{C_d}{r^{(d-1)/2}}\, \mathrm{e}^{\mathrm{i}kr}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}'}\,, \qquad \mathbf{k}' \equiv k\hat{\mathbf{r}}, \tag{92}$$

where C_d is a constant depending only on d. For $d = 3$: $G_+(\mathbf{r}, \mathbf{r}') = -\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}/(4\pi|\mathbf{r} - \mathbf{r}'|)$, which for $|\mathbf{r}| \gg |\mathbf{r}'|$ is approximately $-\mathrm{e}^{\mathrm{i}kr}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}'}/(4\pi r)$, recovering (92) with $\alpha = 1 = (d-1)/2$.

The key property is that the exponent in r is $1/r^{(d-1)/2}$ *already inside the Green's function* — it is not a choice imposed *a posteriori*, it is a property of the Helmholtz operator in d dimensions. Substituting (92) into (90):

$$\psi(\mathbf{r}) \xrightarrow{r\to\infty} \mathrm{e}^{\mathrm{i}kz} + \underbrace{\left[-C_d \int \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}'}\, U(r')\,\psi(\mathbf{r}')\,d^d r'\right]}_{\equiv f(\theta)} \frac{\mathrm{e}^{\mathrm{i}kr}}{r^{(d-1)/2}}\,. \tag{93}$$

The integral in brackets depends only on the outgoing direction $\hat{\mathbf{r}}$ (i.e., only on θ by axial symmetry around the beam axis), and by definition is the scattering amplitude $f(\theta)$. The structure $1/r^{(d-1)/2}$ appears automatically.

Conclusion. The boundary condition (94) is not an arbitrary postulate: it is the unique asymptotic form compatible with (i) the presence of an incident plane wave, (ii) probability conservation in d dimensions, and (iii) the choice of outgoing solutions (causality) of the Helmholtz equation. The three arguments coincide because they express the same physical fact in different ways.

Conclusion: the boundary condition is

$$\psi(\mathbf{r}) \xrightarrow{r\to\infty} \mathrm{e}^{\mathrm{i}kz} + f(\theta)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r^{(d-1)/2}}\,, \tag{94}$$

where $z = r\cos\theta$ and $f(\theta)$ is the scattering amplitude. The envelope $r^{-(d-1)/2}$ ensures finite scattered flux through $r^{d-1}\mathrm{d}\Omega_{d-1}$.

8.2 Phase shifts: definition

The reduced radial solution in the exterior ($V \approx 0$) is a combination of the two free solutions. The **phase shift** $\delta_\ell(k)$ is defined by the asymptotic form

$$u_\ell(r) \xrightarrow{r\to\infty} C_\ell \sin\!\left(kr - \frac{\ell\pi}{2} - \frac{(d-3)\pi}{4} + \delta_\ell\right). \tag{95}$$

Here C_ℓ is a *normalisation constant* of the reduced radial function (not to be confused with the Gegenbauer polynomial C_ℓ^ν, which carries the superscript ν): since the radial equation is linear of second order, the asymptotic form is characterised by two parameters — the amplitude C_ℓ and the phase δ_ℓ. All the physical information of the scattering in the partial wave ℓ is contained in δ_ℓ; the constant C_ℓ will be fixed *a posteriori* by matching with the plane-wave expansion (§8.4). For a free particle, $\delta_\ell = 0$ for all ℓ.

8.3 Computing δ_ℓ from a given $V(r)$

The matching procedure involves three steps. First, one solves the reduced equation (58) in the interior ($r < R_0$) with $u_\ell(0) = 0$ and the specific $V(r)$. Second, one computes the logarithmic derivative at $r = R_0$:

$$\gamma_\ell \equiv \frac{u_\ell'(R_0)}{u_\ell(R_0)}\,. \tag{96}$$

Third, one matches with the exterior solution $u_\ell^{\mathrm{ext}}(r) = A[\sqrt{kr}\,J_\lambda(kr)\cos\delta_\ell - \sqrt{kr}\,N_\lambda(kr)\sin\delta_\ell]$ at $r = R_0$. Continuity of u_ℓ'/u_ℓ gives

$$\tan\delta_\ell = \frac{[\sqrt{\rho}\,J_\lambda(\rho)]'\big|_{\rho_0} - \gamma_\ell\sqrt{\rho_0}\,J_\lambda(\rho_0)/k}{[\sqrt{\rho}\,N_\lambda(\rho)]'\big|_{\rho_0} - \gamma_\ell\sqrt{\rho_0}\,N_\lambda(\rho_0)/k}\,, \qquad \rho_0 = kR_0. \tag{97}$$

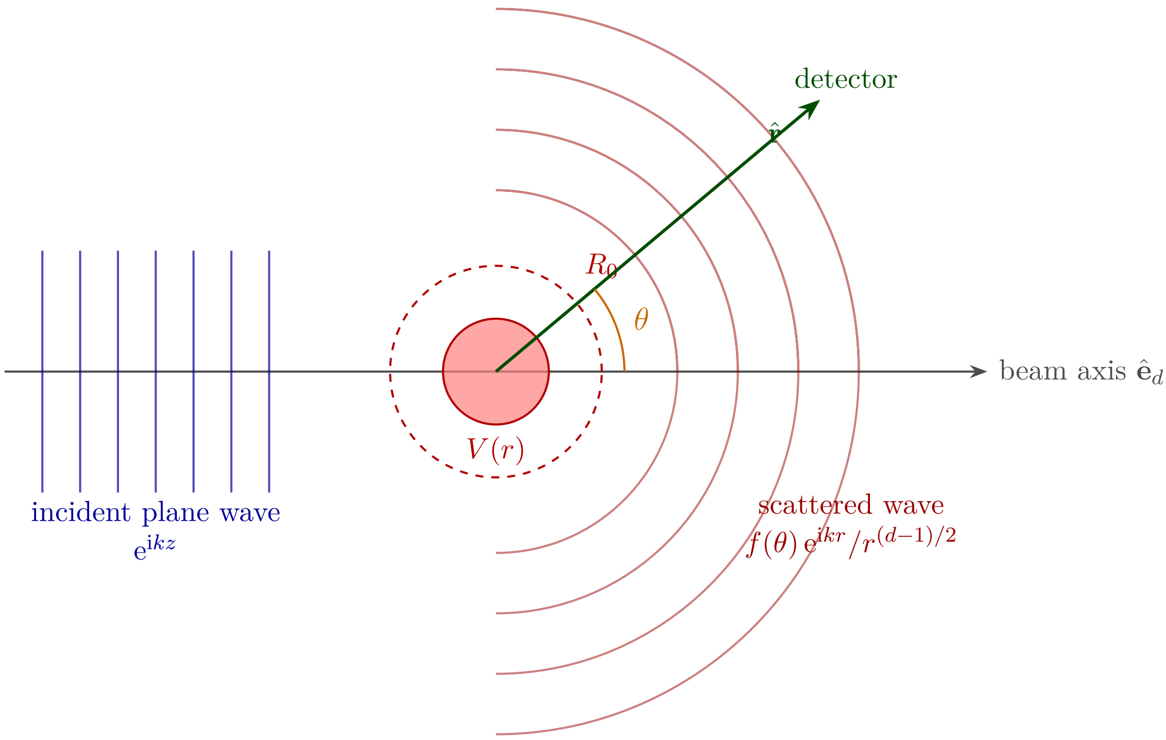

Figure 2: Geometry of the scattering process (scattering plane). By axial symmetry around the beam axis, all the physics happens in a single plane containing $\hat{\mathbf{e}}_d$ and $\hat{\mathbf{r}}$, shown here. The incident plane wave e^{ikz} (blue, vertical wave fronts on the left) propagates along the beam axis with wave vector $\mathbf{k} = k\hat{\mathbf{e}}_d$ and meets a localised potential $V(r)$ (red, central). Beyond the radius R_0 (dashed circle) the potential is negligible by the short-range hypothesis. The interaction produces an outgoing spherical wave $f(\theta)\,e^{ikr}/r^{(d-1)/2}$ (red circular fronts on the right), whose amplitude in each direction is given by the scattering amplitude $f(\theta)$. The **scattering angle** θ (orange) is measured between the beam axis $\hat{\mathbf{e}}_d$ and the detector direction $\hat{\mathbf{r}}$.

8.4 Phase shifts → scattering amplitude: matching

This is the crucial derivation that produces $f(\theta)$ from the δ_ℓ. We detail each step.

Set-up. We have three ingredients:

(i) The full wave function in partial waves:

$$\psi(\mathbf{r}) = \sum_\ell a_\ell \, \frac{u_\ell(r)}{r^{(d-1)/2}} \, C_\ell^\nu(\cos\theta), \tag{98}$$

with $u_\ell \sim C_\ell \sin(kr - \phi_\ell + \delta_\ell)$ at large r, where

$$\phi_\ell \equiv \frac{\ell\pi}{2} + \frac{(d-3)\pi}{4} \,. \tag{99}$$

The product $a_\ell C_\ell$ is unknown.

(ii) The plane wave in partial waves (Rayleigh expansion), with asymptotic form

$$\mathrm{e}^{\mathrm{i}kz} \sim \sum_\ell \Gamma(\nu)(\ell+\nu)\mathrm{i}^\ell \cdot \frac{\mathcal{N}}{(kr)^{\nu+1/2}} \sin(kr - \phi_\ell) \, C_\ell^\nu(\cos\theta), \tag{100}$$

where $\mathcal{N} = 2^\nu\sqrt{2/\pi}$ is the asymptotic constant of (70).

(iii) The scattered wave $f(\theta)\,\mathrm{e}^{\mathrm{i}kr}/r^{(d-1)/2}$ is purely outgoing.

Step 1: Decompose into incoming and outgoing waves. We write each $\sin(X) = (\mathrm{e}^{\mathrm{i}X} - \mathrm{e}^{-\mathrm{i}X})/(2\mathrm{i})$.

For the *full wave* (using $(d-1)/2 = \nu + \frac{1}{2}$):

$$\psi \sim \sum_\ell a_\ell C_\ell \, \frac{C_\ell^\nu(\cos\theta)}{2\mathrm{i}\, r^{\nu+1/2}} \left[\underbrace{\mathrm{e}^{\mathrm{i}(kr-\phi_\ell+\delta_\ell)}}_{\text{outgoing}} - \underbrace{\mathrm{e}^{-\mathrm{i}(kr-\phi_\ell+\delta_\ell)}}_{\text{incoming}} \right]. \tag{101}$$

For the *plane wave*:

$$\mathrm{e}^{\mathrm{i}kz} \sim \sum_\ell \Gamma(\nu)(\ell+\nu)\mathrm{i}^\ell \frac{\mathcal{N}\, C_\ell^\nu(\cos\theta)}{2\mathrm{i}\, k^{\nu+1/2}\, r^{\nu+1/2}} \left[\underbrace{\mathrm{e}^{\mathrm{i}(kr-\phi_\ell)}}_{\text{outgoing}} - \underbrace{\mathrm{e}^{-\mathrm{i}(kr-\phi_\ell)}}_{\text{incoming}} \right]. \tag{102}$$

The *scattered wave* is purely outgoing (only $\mathrm{e}^{+\mathrm{i}kr}$).

Step 2: Match the incoming waves. The boundary condition states that $\psi = \mathrm{e}^{\mathrm{i}kz} + f(\theta)\mathrm{e}^{\mathrm{i}kr}/r^{(d-1)/2}$. Since the scattered wave contains only $\mathrm{e}^{+\mathrm{i}kr}$, the incoming part ($\mathrm{e}^{-\mathrm{i}kr}$) of ψ must come *entirely* from the plane wave. For each ℓ, the incoming coefficients must match.

The incoming coefficient of C_ℓ^ν in the full wave is

$$-\frac{a_\ell C_\ell}{2\mathrm{i}} \, \mathrm{e}^{\mathrm{i}(\phi_\ell-\delta_\ell)} \, \frac{\mathrm{e}^{-\mathrm{i}kr}}{r^{\nu+1/2}} \,, \tag{103}$$

and in the plane wave is

$$-\frac{\Gamma(\nu)(\ell+\nu)\mathrm{i}^\ell \mathcal{N}}{2\mathrm{i}\, k^{\nu+1/2}} \, \mathrm{e}^{\mathrm{i}\phi_\ell} \, \frac{\mathrm{e}^{-\mathrm{i}kr}}{r^{\nu+1/2}} \,. \tag{104}$$

Equating and cancelling common factors:

$$a_\ell C_\ell \, \mathrm{e}^{-\mathrm{i}\delta_\ell} = \frac{\Gamma(\nu)(\ell+\nu)\mathrm{i}^\ell \mathcal{N}}{k^{\nu+1/2}}\,, \tag{105}$$

which on isolating $a_\ell C_\ell$ gives

$$a_\ell C_\ell = \frac{\Gamma(\nu)(\ell+\nu)\,\mathrm{i}^\ell \mathcal{N}}{k^{\nu+1/2}}\,\mathrm{e}^{\mathrm{i}\delta_\ell}\,. \tag{106}$$

The phase shift appears as a phase factor. For $\delta_\ell = 0$ (no potential), it reduces to the plane-wave coefficient, as expected.

Step 3: Read off $f(\theta)$ from the outgoing excess. The *outgoing* coefficient of C_ℓ^ν in the full wave is

$$+\frac{a_\ell C_\ell}{2\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}(\phi_\ell-\delta_\ell)} = \frac{\Gamma(\nu)(\ell+\nu)\mathrm{i}^\ell \mathcal{N}}{2\mathrm{i}\,k^{\nu+1/2}}\,\mathrm{e}^{-\mathrm{i}\phi_\ell}\,\mathrm{e}^{2\mathrm{i}\delta_\ell}\,, \tag{107}$$

where we have used (106). The outgoing coefficient of the plane wave alone is the same expression with $\mathrm{e}^{2\mathrm{i}\delta_\ell} \to 1$. The **outgoing excess** is

$$\frac{\Gamma(\nu)(\ell+\nu)\mathrm{i}^\ell \mathcal{N}}{2\mathrm{i}\,k^{\nu+1/2}}\,\mathrm{e}^{-\mathrm{i}\phi_\ell}\,(\mathrm{e}^{2\mathrm{i}\delta_\ell}-1). \tag{108}$$

The boundary condition states that this equals the contribution of the scattered wave $f_\ell\,\mathrm{e}^{\mathrm{i}kr}/r^{\nu+1/2}$. Cancelling $\mathrm{e}^{\mathrm{i}kr}/r^{\nu+1/2}$ and using $\mathrm{i}^\ell = \mathrm{e}^{\mathrm{i}\ell\pi/2}$ together with $\phi_\ell = \ell\pi/2 + (d-3)\pi/4$:

$$\mathrm{i}^\ell\,\mathrm{e}^{-\mathrm{i}\phi_\ell} = \mathrm{e}^{-\mathrm{i}(d-3)\pi/4}\,, \tag{109}$$

a constant independent of ℓ. Absorbing this together with $\mathcal{N} = 2^\nu\sqrt{2/\pi}$ into a single prefactor

$$A_d \equiv \frac{2^\nu\sqrt{2/\pi}\,\Gamma(\nu)\,\mathrm{e}^{-\mathrm{i}(d-3)\pi/4}}{k^{1/2}}, \tag{110}$$

we arrive at the final result:

$$\boxed{f(\theta) = \frac{A_d}{k^\nu}\sum_{\ell=0}^{\infty}(\ell+\nu)\,\frac{\mathrm{e}^{2\mathrm{i}\delta_\ell}-1}{2\mathrm{i}}\,C_\ell^\nu(\cos\theta) = \frac{A_d}{k^\nu}\sum_{\ell=0}^{\infty}(\ell+\nu)\,\mathrm{e}^{\mathrm{i}\delta_\ell}\sin\delta_\ell\,C_\ell^\nu(\cos\theta).} \tag{111}$$

8.5 Physical picture

The plane wave is a superposition of incoming and outgoing spherical waves in each partial wave. The potential shifts the phase of the outgoing part by $2\delta_\ell$; the incoming part is left untouched. The scattered wave is the outgoing excess. For $\delta_\ell = 0$, the factor $\mathrm{e}^{2\mathrm{i}\delta_\ell}-1$ vanishes, and there is no scattering. For $\delta_\ell = \pi/2$, the factor equals $2\mathrm{i}$, attaining the unitarity limit.

8.6 Cross sections

We recall the operational definition of the differential cross section: it is the ratio of the rate of particles scattered per unit solid angle to the incident flux.

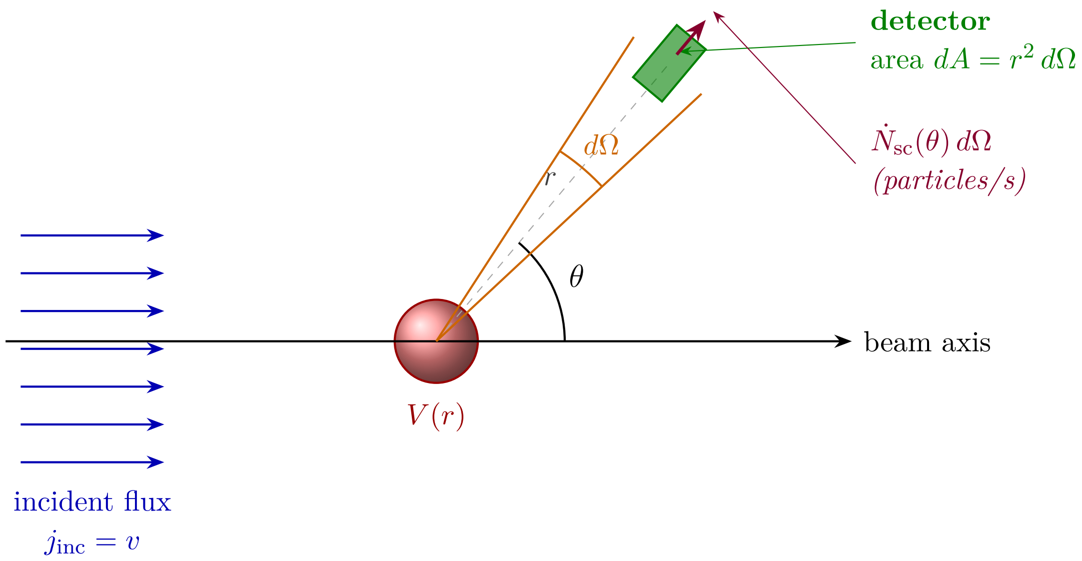

Figure 3: Definition of the differential cross section. An incident beam (blue) with flux $j_{\rm inc}$ crosses the scattering centre (red). A fraction of the particles emerges at angle θ and crosses the detector (green) at distance r, subtending the solid angle $d\Omega$ (orange cone), of area $dA = r^2\,d\Omega$. The rate of particles reaching the detector is $\dot N_{\rm sc}(\theta)\,d\Omega$. The differential cross section is defined as this rate per unit solid angle, divided by the incident flux — an effective area.

Definition 8.1 (Differential cross section)**.** The differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \equiv \frac{\text{(particles scattered per s per solid angle)}}{j_{\rm inc}}\,, \tag{112}$$

with dimensions $[\text{length}]^{d-1}$.

To derive $d\sigma/d\Omega = |f(\theta)|^2$, we compute the two ingredients of Figure 3. For the plane wave $\psi_{\rm inc} = \mathrm{e}^{\mathrm{i}kz}$, the probability current is $\mathbf{j}_{\rm inc} = (\hbar/m)\,\mathrm{Im}[\psi^*\nabla\psi] = (\hbar k/m)\hat{\mathbf{e}}_d = v\,\hat{\mathbf{e}}_d$, and $j_{\rm inc} = v$. For $\psi_{\rm sc} \sim f(\theta)\mathrm{e}^{\mathrm{i}kr}/r^{(d-1)/2}$, the radial component of the current at the detector is $j_r^{\rm sc} = v\,|f(\theta)|^2/r^{d-1}$, and the rate of particles through $dA = r^{d-1}d\Omega$ is $\dot N_{\rm sc}\,d\Omega = v\,|f(\theta)|^2\,d\Omega$. The r^{d-1} from the area cancels exactly the $r^{-(d-1)}$ from $|\psi_{\rm sc}|^2$ — the reason the envelope was chosen $1/r^{(d-1)/2}$. The final ratio gives $d\sigma/d\Omega = |f(\theta)|^2$.

The total cross section, using Gegenbauer orthogonality, is

$$\sigma_{\rm tot} = \frac{\omega_{d-1}|A_d|^2}{k^{2\nu}}\sum_{\ell=0}^{\infty}(\ell+\nu)^2\sin^2\delta_\ell\,h_\ell^{(\nu)}\,, \tag{113}$$

and the optical theorem gives $\sigma_{\rm tot} \propto \mathrm{Im}[f(0)]/k^\nu$.

8.7 Low-energy limit

For a finite-range potential,

$$\delta_\ell(k) \sim k^{2\lambda} = k^{2\ell+d-2} \qquad \text{as } k \to 0. \tag{114}$$

At low energy, only $\ell = 0$ survives (for $d \geq 3$). We define the **scattering length**

$$a_d^{d-2} \equiv -\lim_{k\to 0}\frac{\tan\delta_0}{k^{d-2}}\,, \tag{115}$$

with the effective-range expansion (for $d \geq 3$) $k^{d-2}\cot\delta_0 = -1/a_d^{d-2} + \frac{1}{2}r_d\,k^2 + \mathcal{O}(k^4)$. For $d = 2$, $\lambda_0 = 0$ and the threshold behaviour is logarithmic ($\delta_0 \sim -1/\ln ka$), requiring separate treatment.

Part II
Specific Potentials

We now apply the procedure to several potentials.

9 Hard Sphere: $V(r) = \infty$ for $r < a$

The hard wall imposes $u_\ell(a) = 0$. The exterior solution satisfies this condition when

$$\tan\delta_\ell = \frac{J_\lambda(ka)}{N_\lambda(ka)}\,. \tag{116}$$

At low energy, using (71): $\tan\delta_\ell \sim (ka)^{2\lambda} = (ka)^{2\ell+d-2}$, confirming (114). The s-wave ($\ell = 0$, $\lambda = \nu$) gives $\tan\delta_0 = J_\nu(ka)/N_\nu(ka)$. For $d = 3$, this yields the exact $\delta_0 = -ka$, $a_{\rm sc} = a$ and $\sigma_{\rm tot} \to 4\pi a^2$ — four times the classical geometric cross section, a quantum diffraction effect.

10 Finite Spherical Well: $V(r) = -V_0$ for $r < a$

In the interior, the radial equation becomes the free equation with $k \to K = \sqrt{k^2 + 2mV_0/\hbar^2}$. The regular solution is

$$u_\ell^{\rm in}(r) = B\sqrt{Kr}\,J_\lambda(Kr). \tag{117}$$

Continuity of u_ℓ and u_ℓ' at $r = a$ gives

$$\tan\delta_\ell = \frac{k\,[\sqrt{\rho}\,J_\lambda]'|_{ka}\,J_\lambda(Ka) - K\,[\sqrt{\rho}\,J_\lambda]'|_{Ka}\,J_\lambda(ka)}{k\,[\sqrt{\rho}\,N_\lambda]'|_{ka}\,J_\lambda(Ka) - K\,[\sqrt{\rho}\,J_\lambda]'|_{Ka}\,N_\lambda(ka)}\,. \tag{118}$$

When δ_ℓ passes through $\pi/2 \pmod{\pi}$, $\sin^2\delta_\ell = 1$ (unitarity limit). Near a resonance at E_r with width Γ:

$$\sigma_\ell \propto \frac{(\Gamma/2)^2}{(E - E_r)^2 + (\Gamma/2)^2} \qquad \text{(Breit–Wigner)}, \tag{119}$$

physically corresponding to temporary trapping with lifetime $\tau = \hbar/\Gamma$. The s-wave scattering length diverges ($a_d \to \pm\infty$) precisely when the potential is on the verge of supporting a new bound state — a deep and universal connection.

11 Born Approximation and the Yukawa Potential

11.1 The Born approximation

When the potential is weak or the energy is high, the wave function inside the interaction region is approximately the incident plane wave itself. Substituting $\psi(\mathbf{r}')$ by $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}$ in the exact integral expression for f:

$$f^{(1)}(\theta) \propto -\frac{2m}{\hbar^2}\int \mathrm{d}^d r'\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}'} V(r'), \qquad \mathbf{q} = \mathbf{k}' - \mathbf{k}, \quad q = 2k\sin\frac{\theta}{2}\,. \tag{120}$$

For a central potential, the angular integration gives

$$f^{(1)}(\theta) \propto -\frac{2m}{\hbar^2}\int_0^\infty r'^{d-1} V(r')\frac{J_\nu(qr')}{(qr')^\nu}\mathrm{d}r'\,. \tag{121}$$

The angular integral is the $\ell = 0$ case of the Rayleigh expansion (73): all other contributions vanish by Gegenbauer orthogonality. The Born approximation is accurate when $|V| \ll E$ or $|V| \ll \hbar^2/(ma^2)$.

11.2 The Yukawa potential

Definition 11.1 (Yukawa potential in d dimensions)**.**

$$V(r) = V_0\,\frac{\mathrm{e}^{-\mu r}}{r}\,, \tag{122}$$

with V_0 the coupling constant and $\mu > 0$ the inverse range. The definition is a radial function; it has the same functional form in every dimension.

The Yukawa potential satisfies all the standing assumptions in every d. It describes the interaction mediated by massive bosons: pion exchange between nucleons ($1/\mu \sim 1.4\,\mathrm{fm}$), screened Coulomb in plasmas (with $1/\mu$ the Debye length), and the weak force at low energies.

11.3 Born amplitude in d dimensions

Substituting (122) into (121):

$$f^{(1)}(\theta) \propto -\frac{2mV_0}{\hbar^2}\int_0^\infty r'^{d-2}\,\mathrm{e}^{-\mu r'}\,\frac{J_\nu(qr')}{(qr')^\nu}\,\mathrm{d}r'\,. \tag{123}$$

For general d the integral is not elementary (it can be written in terms of ${}_2F_1$), but the physical content is clear: the Born amplitude is the d-dimensional Fourier transform of a short-range potential and decays as q increases. The exponential cut-off $\mathrm{e}^{-\mu r}$ ensures convergence in every d.

11.4 Specialisation to $d = 3$

For $d = 3$, $\nu = \frac{1}{2}$, and $J_{1/2}(z)/z^{1/2} = \sqrt{2/\pi}\,\sin(z)/z$. Substituting:

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2}\int_0^\infty \mathrm{e}^{-\mu r'}\frac{\sin(qr')}{q}\,\mathrm{d}r' \qquad [d=3]\,. \tag{124}$$

The integral is elementary. Writing $\sin(qr') = (\mathrm{e}^{\mathrm{i}qr'} - \mathrm{e}^{-\mathrm{i}qr'})/(2\mathrm{i})$:

$$\int_0^\infty \mathrm{e}^{-\mu r'}\sin(qr')\mathrm{d}r' = \frac{q}{\mu^2+q^2}\,, \tag{125}$$

hence $\int_0^\infty \mathrm{e}^{-\mu r'} \sin(qr')/q \, \mathrm{d}r' = 1/(q^2+\mu^2)$. The final result in $d = 3$ is

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2} \cdot \frac{1}{q^2+\mu^2} = -\frac{2mV_0/\hbar^2}{\mu^2 + 4k^2 \sin^2(\theta/2)} \qquad [d=3]\,, \tag{126}$$

with differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f^{(1)}|^2 = \left(\frac{2mV_0}{\hbar^2}\right)^2 \frac{1}{[\mu^2 + 4k^2 \sin^2(\theta/2)]^2} \qquad [d=3]\,. \tag{127}$$

11.5 Three important limits

Coulomb limit: $\mu \to 0$. Taking $\mu \to 0$ with $V_0 \equiv \alpha$ (so that $V(r) \to \alpha/r$):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \xrightarrow{\mu\to 0} \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}\,. \tag{128}$$

This is the Rutherford formula! The Born approximation for the Yukawa potential, in the zero-mass limit of the mediator, reproduces the exact Coulomb cross section. Remarkably, the Rutherford formula is identical in three distinct treatments: classical (Rutherford, 1911), Born (above) and exact quantum (via separation in parabolic coordinates). This agreement is specific to the $1/r$ potential and is related to its hidden $SO(4)$ symmetry — the same one that gives the accidental degeneracy of hydrogen.

Low-energy limit: $k \to 0$. For $k \ll \mu$, $f^{(1)} \to -2mV_0/(\hbar^2\mu^2) \equiv -a_Y$, isotropic. Identifying $-a_Y$ with the scattering length:

$$a_Y^{(\mathrm{Born})} = \frac{2mV_0}{\hbar^2\mu^2}\,, \tag{129}$$

and $\sigma_{\mathrm{tot}} \to 4\pi a_Y^2$.

High-energy limit: $k \gg \mu$. For $q \gg \mu$, μ^2 is negligible compared with q^2:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \xrightarrow{k\gg\mu} \left(\frac{2mV_0}{\hbar^2}\right)^2 \frac{1}{q^4}\,, \tag{130}$$

the screened Rutherford formula: at high energies, the particle probes distances much smaller than $1/\mu$ and sees an effective $1/r$.

11.6 An additional remark on the electrostatic Yukawa

The potential (122) retains the same functional form $\mathrm{e}^{-\mu r}/r$ in every d. A different generalisation, physically motivated, is the screened Poisson Green's function $G_\mu^{(d)}(r) \propto (\mu/r)^\nu K_\nu(\mu r)$, the solution of $(-\nabla_d^2 + \mu^2)G = \delta^{(d)}(\mathbf{r})$, whose d-dim Fourier transform is $1/(q^2+\mu^2)$ in every d. For $d = 3$, $K_{1/2}(z) = \sqrt{\pi/(2z)}\,\mathrm{e}^{-z}$ and $G_\mu^{(3)} \propto \mathrm{e}^{-\mu r}/r$ — the usual Yukawa. For $d \neq 3$ the two generalisations differ.

12 Coulomb-type Potentials

There are two natural candidates for the "Coulomb potential" in d dimensions, and they differ for $d \neq 3$. We treat each separately.

Option A is $V(r) = -\alpha/r$ in every d. This is not the electrostatic potential of a point charge for $d \neq 3$, but has the special mathematical property that the radial equation reduces to the confluent hypergeometric equation in every d.

Option B is $V(r) = -\alpha/r^{d-2}$. This is the genuine electrostatic potential in d dimensions (the Green's function of ∇_d^2). But for $d \neq 3$ it does not reduce to a standard special-function equation.

To see why the electrostatic potential is $1/r^{d-2}$, one applies Gauss's law in d dimensions: $E(r) \cdot \Omega_d \, r^{d-1} = \text{const}$, hence $E(r) \propto 1/r^{d-1}$. Integrating $V = -\int E \mathrm{d}r$ gives $V(r) \propto 1/r^{d-2}$. For $d = 3$ this is $1/r$; for $d = 4$ it is $1/r^2$; for $d = 2$ it is $\ln r$.

12.1 Option A: $V = -\alpha/r$ — the confluent hypergeometric equation

For $V(r) = -\alpha/r$, the radial equation becomes

$$u_\ell'' + \left[k^2 + \frac{2\eta k}{r} - \frac{\lambda^2 - \frac{1}{4}}{r^2}\right] u_\ell = 0, \qquad \eta = \frac{m\alpha}{\hbar^2 k}, \tag{131}$$

with η the Sommerfeld parameter. The substitution $\rho = -2\mathrm{i}kr$ transforms it into the confluent hypergeometric equation

$$\rho\, w'' + (b - \rho)\, w' - a\, w = 0, \tag{132}$$

with $a = \lambda + \frac{1}{2} + \mathrm{i}\eta = \ell + (d-1)/2 + \mathrm{i}\eta$ and $b = 2\lambda + 1 = 2\ell + d - 1$. The regular solution is

$$u_\ell(r) = C\,(2kr)^{\lambda+1/2}\, \mathrm{e}^{\mathrm{i}kr}\, {}_1F_1\big(\lambda + \tfrac{1}{2} + \mathrm{i}\eta;\ 2\lambda+1;\ -2\mathrm{i}kr\big). \tag{133}$$

This is the same ${}_1F_1$ that appears in the bound states of the hydrogen atom.

The asymptotic expansion for large $|z|$ of ${}_1F_1(a;b;z)$ is

$${}_1F_1(a;b;z) \sim \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a} + \frac{\Gamma(b)}{\Gamma(a)}\mathrm{e}^{z} z^{a-b}\,. \tag{134}$$

For $z = -2\mathrm{i}kr$, these terms oscillate as $\mathrm{e}^{\mp \mathrm{i}kr}$ (incoming and outgoing waves). The ratio of outgoing to incoming amplitudes gives the S-matrix element. Since $|\Gamma(\alpha + \mathrm{i}\beta)| = |\Gamma(\alpha - \mathrm{i}\beta)|$, this ratio has unit modulus (unitarity), and its phase gives

$$S_\ell^C = \mathrm{e}^{2\mathrm{i}\sigma_\ell^C}, \qquad \sigma_\ell^C = \arg\Gamma\big(\lambda + \tfrac{1}{2} + \mathrm{i}\eta\big). \tag{135}$$

In $d = 3$, the exact amplitude can be obtained from the separation in parabolic coordinates:

$$f_C(\theta) = -\frac{\eta}{2k\sin^2(\theta/2)}\exp\left[-\mathrm{i}\eta\ln\sin^2\frac{\theta}{2} + 2\mathrm{i}\sigma_0\right], \tag{136}$$

giving the Rutherford cross section $d\sigma/d\Omega = \eta^2/[4k^2\sin^4(\theta/2)]$. The $1/r$ tail modifies the asymptotic wave function at all distances, producing a logarithmic phase that prevents pointwise convergence of the partial-wave series; the sum requires Sommerfeld regularisation.

12.2 Option B: $V = -\alpha/r^{d-2}$ — the genuine electrostatic potential

For the genuine d-dimensional Coulomb potential, the radial equation is

$$u_\ell'' + \left[k^2 - \frac{\tilde{\alpha}}{r^{d-2}} - \frac{\lambda^2 - \frac{1}{4}}{r^2}\right] u_\ell = 0, \qquad \tilde{\alpha} = \frac{2m\alpha}{\hbar^2}\,. \tag{137}$$

For $d = 3$ we recover (131). For $d \neq 3$ the structure changes qualitatively: in $d = 2$, $V \propto \ln r$ (logarithmic); in $d = 4$, $V \propto 1/r^2$, merging with the centrifugal barrier and producing pathologies (fall to the centre, conformal symmetry); in $d = 5$, $V \propto 1/r^3$; in general, with no closed-form solution in standard special functions. The Born approximation for $V = -\alpha/r^{d-2}$ works in every d and generalises Rutherford, since the Fourier transform of $1/r^{d-2}$ in d dim is $1/q^2$ (up to a constant).

13 Summary

The central points of the document can be summarised as follows.

Structure. The d-dimensional problem of scattering by a central potential separates into a universal angular equation (valid for every V) and a potential-specific radial equation. The angular one has eigenvalue $\ell(\ell + d - 2)$ and eigenfunctions in hyperspherical form involving Gegenbauer polynomials C_ℓ^ν with $\nu = (d - 2)/2$. The radial one reduces to an effective 1D problem via $R = u/r^{(d-1)/2}$, with effective angular momentum $\mathcal{L} = \ell + (d - 3)/2$.

Procedure. Given $V(r)$: (i) solve the radial equation in the interior; (ii) compute the logarithmic derivative γ_ℓ at R_0; (iii) match with the exterior solution to obtain δ_ℓ; (iv) sum the partial-wave expansion to get $f(\theta)$; (v) the cross section is $|f|^2$, and at low energy one extracts the scattering length a_d.

Substitution rule $d = 3 \to d$. $\ell(\ell+1) \to \ell(\ell+d-2)$, $j_\ell, n_\ell \to J_{\ell+\nu}, N_{\ell+\nu}$, $P_\ell \to C_\ell^\nu$, $r^{-1} \to r^{-(d-1)/2}$.

Examples. The hard sphere in $d = 3$ gives $\sigma_{\text{tot}} \to 4\pi a^2$ at low energy (four times the classical value). The spherical well exhibits Breit–Wigner-type resonances. The Born approximation applied to the Yukawa potential gives $f^{(1)} \propto -1/(q^2 + \mu^2)$ in $d = 3$ and, in the limit $\mu \to 0$, recovers Rutherford.

Pathological case of Coulomb. The $1/r$ potential has infinite range, violates the short-range hypothesis, and requires separate treatment — with Coulomb wave functions and separation in parabolic coordinates. Apart from this, Option A ($1/r$ in every d) reduces to the confluent hypergeometric equation, the same one as for hydrogen: bound states ($E < 0$) and scattering ($E > 0$) are two faces of the same mathematical problem.

14 The complete scattering procedure

$$V(r) \xrightarrow[\text{solve radial eq.}]{\text{interior}} \gamma_\ell \xrightarrow[\text{Eq. (97)}]{\text{matching at } R_0} \delta_\ell(k) \xrightarrow[\text{Eq. (111)}]{\text{Gegenbauer sum}} f(\theta) \xrightarrow{|f|^2} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \xrightarrow{k\to 0} a_d$$