Riemann Zeta function

Riemann Zeta func: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

for $Re(s) > 1$.

The function converges absolutely and uniformly on compact subsets of $\{Re(s) > 1\}$, as

$|n^s| = n^{Re(s)}$.

$\zeta(s)$ is holomorphic function

on $Re(z) > 1$.

Connection with primes

Euler product formula: for $Re(s) > 1$, the infinite product $\prod_{p \in P} \left(1 - \frac{1}{p^s}\right)$ converges and

$$\frac{1}{\zeta(s)} = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right),$$

where P is the set of positive primes $\{2, 3, 5, 7, \dots\}$

Proof: Since $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $Re\, s > 1$, the prod $\prod_{p \in P} \left(1 - \frac{1}{p^s}\right)$ also converges.

Fix s with $\operatorname{Re} s > 1$ & $\varepsilon > 0$.

Let's choose N so large that

$$\sum_{n=N+1}^{\infty} \left| \frac{1}{n^s} \right| < \varepsilon$$

Now $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \dots$$

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \dots$$

This is a well-known number-theoretic procedure known as sieve of Eratosthenes.

Thus: $\left(1-\frac{1}{(p_N)^s}\right)\cdot\left(1-\frac{1}{(p_{N-1})^s}\right)\cdots$

$$\left(1-\frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{(p_{N+1})^s} + \cdots$$

By the choice of N, if $n \geq N$ then:

$$\left|\left(\prod_{j=1}^{n}\left(1-\frac{1}{(p_j)^s}\right)\right)\zeta(s) - 1\right| < \varepsilon,$$

as desired. Completing the proof that $\frac{1}{\zeta(s)} = \prod_{p\in P}\left(1-\frac{1}{p^s}\right)$

Relation to the gamma function

for $\mathrm{Re}(z) > 1$

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty dt \, \frac{t^{z-1} e^{-t}}{1 - e^{-t}}$$

proof: The integral converges for the same reason that the gamma function converges. For $j = 1, 2, \ldots$

$$j^{-z} \int_0^\infty \tau^{z-1} e^{-\tau} d\tau \overset{(jt = \tau)}{=} \int_0^\infty t^{z-1} e^{-jt} dt$$

or

$$j^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-jt} dt$$

Summing over j gives

$$\zeta(z) = \sum_{j=1}^\infty j^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty dt \, t^{z-1} \underbrace{\sum_{j=1}^\infty e^{-jt}}_{= \frac{e^{-t}}{1 - e^{-t}}}$$

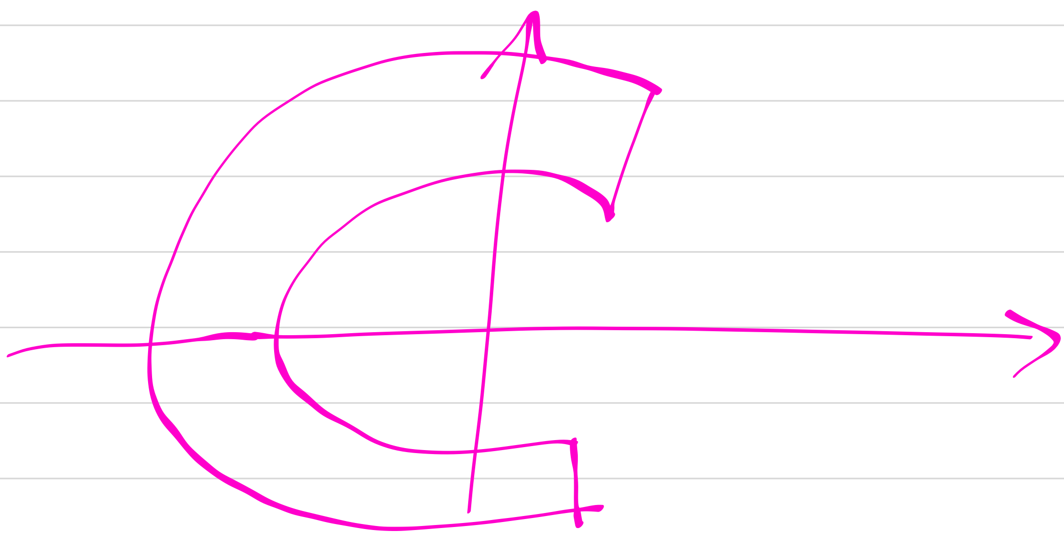
Thus: $\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty dt \, \frac{t^{z-1} e^{-t}}{1 - e^{-t}}$. //

For $z \in \mathbb{C}$, define the aux function $u(w) = \dfrac{(-w)^{z-1} e^{-w}}{1-e^{-w}}$

On the region $\mathbb{C} \setminus \{w : \operatorname{Re} w \geq 0, \operatorname{Im} w = 0\}$ the func. is well defined if we take $-\pi < \arg(-w) < \pi$. Also define, for $0 < \epsilon \neq 2\pi k$, Hankel funct

$$H_\epsilon(z) = \int_{C_\epsilon} u(w)\, dw$$

where $C_\epsilon = C_\epsilon(\delta)$ is the Hankel contour shown bellow



The interpretation is the following: linear portions of the contour are understood to lie just above and just bellow the real axis (at distance δ), while the circular portion is understood to have radius ε.

Notice that, for $0 < \varepsilon_1 < \varepsilon_2 < 2\pi$

$H_{\varepsilon_1}(z) = H_{\varepsilon_2}(z)$ since the region bounded by C_{ε_1}, C_{ε_2} contains no poles of u.

For $0 < \varepsilon < 2\pi$ e $\operatorname{Re}(z) > 1$ we have

$$\zeta(z) = -\frac{H_\varepsilon(z)}{2i\sin(\pi z)} \frac{1}{\Gamma(z)}$$

As a result, $\zeta(z)$ continues analytically to $\mathbb{C} \setminus \{1\}$

<u>proof:</u> Parametrizing C_ε :

$$H_\varepsilon = \int_\infty^{\tilde{\varepsilon}} dt \, \left(1-e^{-(t+i\delta)}\right)^{-1} \times \exp\left((z-1)\log\left(-(t+i\delta)\right) - (t+i\delta)\right)$$

$$+ \int_{\tilde{\varepsilon}}^{\infty} \underset{\text{integrand}}{(\text{same})} \, dt$$

$$+ \int_{\tilde{\delta}}^{2\pi-\tilde{\delta}} \frac{i\varepsilon e^{i\theta}}{1-e^{-\varepsilon e^{i\theta}}} \left(-\varepsilon e^{i\theta}\right)^{z-1} \times e^{-\varepsilon e^{i\theta}} = I + II + III$$

Here $\tilde{\delta}$ represents the radian measure of the initial point of the circular portion of the curve C_ε and $\tilde{\varepsilon}$ represents the value of the parameter at which the linear portion of the curve C_ε meets the circular portion.

Now, for small ε,

$$\left|1-e^{-\varepsilon e^{i\theta}}\right| \geq \left|1-e^{-\varepsilon}\right|$$

$$\geq \varepsilon/2, \text{ thus}$$

$$|III| \leq 2\pi \max_{\theta} \left|(-\varepsilon e^{i\theta})^{\beta-1}\right| \times \left|e^{-\varepsilon e^{i\theta}}\right| \frac{\varepsilon}{\varepsilon/2}$$

$$\leq 4\pi \varepsilon^{\operatorname{Re} z - 1} e^{-\theta \operatorname{Im} z} e^{\varepsilon} \to 0$$

as $\varepsilon \to 0$.

On the other hand, $I + II =$

$$\int_{\infty}^{\tilde{\varepsilon}} dt \,(1 - e^{-t - i\delta})^{-1} \exp\{(z-1)$$

$$\times \left[\log\sqrt{t^2+\delta^2} + i(\delta' - \pi)\right] - t - i\delta\}$$

$$+ \int_{\tilde{\varepsilon}}^{\infty} dt \,(1 - e^{-t + i\delta})^{-1} \exp\{(z-1)$$

$$\times \left[\log\sqrt{t^2+\delta^2} + i(\pi - \delta'')\right] - t + i\delta\}$$

where $\delta' = \delta'(t)$ and $\delta'' = \delta''(t)$

are chosen so that $(\delta' - \pi)$, $(\pi - \delta'')$

are the args. of the initial and

terminal points on the curve C_ε.

Note that the value of $H_\varepsilon(z)$ is independent of $\delta > 0$ as long as δ is small. By uniform convergence we may let $\delta \to 0^+$ so that we may restrict ourselves to

$$\int_\infty^{\tilde{\varepsilon}} dt\,(1-e^{-t})^{-1} \exp\{(z-1)\log t - i\pi - t\}$$

$$+\int_{\tilde{\varepsilon}}^{+\infty} dt\,(1-e^{-t})^{-1} \exp\{(z-1)\log t + i\pi - t\}$$

$$= \underbrace{-\left(e^{i\pi z} - e^{-i\pi z}\right)}_{-2i\sin(\pi z)} \int_{\tilde{\varepsilon}}^{\infty} dt\, \frac{t^{z-1}e^{-t}}{1-e^{-t}}$$

As $\tilde{\varepsilon} \to 0^+$, the integral becomes $\Gamma(z)\zeta(z)$. Thus, we have shown that $H(z) \equiv \lim_{\varepsilon \to 0^+} H_\varepsilon(z) = -2i\sin(\pi z)\Gamma(z)\zeta(z)$

For the remaining, notice that for w on positive Real axis and $w>1$

$$|u(w)| \leq A w^{\operatorname{Re} z - 1} e^{-w}, \quad w/ \ A = A(z)$$

Thus, u will be integrable over C_ε for any $z \in \mathbb{C}$ and H_ε will define an entire function of z by diff. under integral sign. Thus, $\zeta(z) = \dfrac{-H_\varepsilon(z)}{2i\sin(\pi z)\Gamma(z)}$

defines an analytic continuation of $\zeta(z)$ to $\mathbb{C} \setminus \mathbb{Z}$. However, $\zeta(z)$ is holomorphic at $z = 2, 3, 4, \ldots$ and on $\{z : \operatorname{Re} z > 1\}$. Moreover, the simple poles of $\Gamma(z)$ at $z = 0, -1, -2, \ldots$ cancel the simple zeros of $\sin \pi z$ at these values.

Therefore, the Riemann removable singularities theorem implies that the denominator continues holomorphically to a nonvanishing function on $\{z : \operatorname{Re} z > 3/2\}$, except for a zero at $z=1$. In conclusion, $\zeta(z)$ continues holomorphically to $\mathbb{C} \setminus \{1\}$.

Simple pole at $z=1$

Let us now demonstrate that $\zeta(z)$ has a simple pole at $z=1$ with residue 1.

For $z=1$, $\sin \pi z = 0$, so that

$$I + II = 0 \qquad \text{and}$$

$$III = \int_0^{2\pi} d\theta \, \frac{e^{-\varepsilon e^{i\theta}}}{1 - e^{-\varepsilon e^{i\theta}}} \, i\varepsilon e^{i\theta}$$

$$= \int_0^{2\pi} d\theta \, \frac{i\varepsilon e^{i\theta}}{e^{\varepsilon e^{i\theta}} - 1}$$

$$= \int_0^{2\pi} d\theta \, \frac{i\varepsilon e^{i\theta}}{(1+\varepsilon e^{i\theta}+R)-1}, \quad |R| \le C \cdot \varepsilon^2$$

$$= \int_0^{2\pi} d\theta \, \frac{i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}+R} \overset{\varepsilon \to 0^+}{=} 2\pi i$$

As a result:

$$\lim_{z \to 1} (z-1)\,\zeta(z) = \lim_{z \to 1} \frac{-H_\varepsilon(z)}{\Gamma(z)} \, \frac{z-1}{2i\sin(\pi z)}$$

$$= \frac{-2\pi i}{1} \, \frac{1}{-2\pi i} = 1.$$

q.e.d.

Functional Relation

We keep motivating ourselves to acquire more explicit information on the continuation of $\zeta(z)$ to $\mathbb{C} \setminus \{1\}$. In keeping on this direction, let us derive the functional relation

$$\zeta(1-z) = 2\,\zeta(z)\,\Gamma(z)\cos\left(\frac{\pi}{2}z\right)(2\pi)^{-z}$$

We start by considering $z \in \{0, -1, -2, \ldots\}$.

Suppose $\operatorname{Re} z > 0$. Let $0 < \varepsilon < 2\pi$ and let n be a positive integer. The idea is to relate $H_\varepsilon(z)$ to $H_{(2n+1)\pi}(z)$ using the calculus of residues and then let $n \to \infty$. Thus,

$$\frac{1}{2\pi i}\left[H_{(2n+1)\pi}(z) - H_\varepsilon(z)\right]$$

is the sum of residues of u in the region $R_{\varepsilon, n} = \{z : \varepsilon < |z| < (2n+1)\pi\}$. The poles of u are simple: $\pm 2n\pi i$.

The residue at $\pm 2\pi k i$ $(k > 0)$ is

$$\lim_{w \to \pm 2k\pi i} (w \mp 2k\pi i) \cdot \frac{(-w)^{z-1}\,e^{-w}}{1 - e^{-w}}$$

But $(\omega \mp 2k\pi i) \dfrac{e^{-\omega}}{1-e^{-\omega}} \longrightarrow 1$

so the residue of u at $\pm 2\pi k i$ is

$$\lim_{\omega \to \pm 2\pi k i} (-\omega)^{z-1} = (\mp 2\pi k i)^{z-1}$$

where we continue to use the principal branch of the logarithm. Hence, we have

$$\exp\left((z-1)\log(\mp 2\pi k i)\right)$$

$$= \exp\left((z-1)\left(\log 2k\pi \mp \frac{i\pi}{2}\right)\right)$$

$$= \exp\left(\mp i(z-1)\frac{\pi}{2}\right)(2k\pi)^{z-1}$$

In summary,

$$H_{(2n+1)\pi}(z) - H_\epsilon(z) = 2\pi i \sum \text{of residues of } u \text{ in } R_{\epsilon, n}$$

$$= 4\pi i \cos \frac{\pi}{2}(z-1) \sum_{k=1}^{n} (2\pi k)^{z-1}$$

Since

$$|u(w)| \le A(z)\,\frac{|w|^{\operatorname{Re} z-1}\, e^{-\operatorname{Re} w}}{|1-e^{-w}|}.$$

So, as $\operatorname{Re} z > 0$:

$$-H_\varepsilon(z) = 4\pi i \cos\left(\frac{\pi}{2}(z-1)\right)(2\pi)^{z-1}\sum_{k=1}^{k} k^{z-1}$$

$$= 4\pi i\,(2\pi)^{z-1}\sin\left(\frac{\pi}{2}z\right)\zeta(1-z)$$

Therefore, $$\zeta(z) = \frac{-H\varepsilon(z)}{2i\sin(\pi z)}\,\frac{1}{\Gamma(z)}$$

$$= (2\pi)^{z}\,\frac{\sin\left(\frac{\pi}{2}z\right)\zeta(1-z)}{\sin(\pi z)\,\Gamma(z)}$$

$$= (2\pi)^{z}\,\frac{1}{2\cos\left(\frac{\pi}{2}z\right)}\,\frac{\zeta(1-z)}{\Gamma(z)}$$

q.e.d.

The result follows for all z by analytic continuation.

Finally, we observe that the functional equation provides us with an explicit way to extend the definition of $\zeta(z)$ to all $\mathbb{C} \setminus \{1\}$.

Additionally, we can of the functional equation as a Reflection formula, giving the values to the left of $\operatorname{Re} z = 1/2$ in terms of those to the right.

Another interesting relation

We can think of the Euler product formula the following: it tells us that, for $\operatorname{Re} z > 1$, $\zeta(z)$ does not vanish. Since $\Gamma(z)$ never vanishes, the functional equation says that $\zeta(1-z)$ can vanish for $\operatorname{Re} z > 1$ only at the zeros of $\cos\left(\frac{\pi}{2} z\right)$, i.e. $z = 1, 3, 5, \ldots$

Therefore, we proceed to show that the only zeros of $\zeta(z)$ not in the set $\{z : 0 \leq \operatorname{Re} z \leq 1\}$ are $z = -2, -4, -6, \ldots$

Let us start with the fact that, if we use the functional relation to calculate $\lim_{z \to 1} \zeta(1-z)$, we observe that the simple pole of $\zeta(z)$ at $z = 1$ cancels the simple zero of the function $\cos\left(\frac{\pi}{2} z\right)$, so the function $\zeta(1-z)$

has no zero at $z=1$. However, for $z=3,5,\dots$ the right-hand side of the functional eq. is a product of many finite-valued nonvanishing factors with $\cos(\frac{\pi}{2}z)$ so that $\zeta(1-z)=0$.

Since all the zeros of $\zeta(z)$, except those at $z=(-2)n$, $n=1,2,\dots$, are in the strip $\{z: 0\le \operatorname{Re} z\le 1\}$, it is known as the critical strip. In connection with this, the celebrated Riemann Hypothesis states that all nontrivial zeros of $\zeta(z)$, i.e. $z\ne(-2)n$, $n=1,2,\dots$, lie on the critical strip $\{z: \operatorname{Re} z=\frac{1}{2}\}$!!!

Let us define die function

$$\Lambda : \{ n \in \mathbb{Z} : n > 0 \} \to \mathbb{R}$$

by $\Lambda(n) = \begin{cases} \log p, & n = p^k, p \in P, 0 < k \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$

For $\operatorname{Re} z > 1$:

$$\sum_{n \geq 2}' \Lambda(n) e^{-z \log n} = \frac{-\zeta'(z)}{\zeta(z)}$$

In order to realize that, let us first note that logarithmic differentiation of the Euler product formula yields

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p \in P}' \frac{(1 - e^{-z \log p})'}{(1 - e^{-z \log p})}$$

$$= \sum_{p \in P} \frac{\log p \, e^{-z \log p}}{1 - e^{-z \log p}}$$

We can also expand $\dfrac{e^{-z\log p}}{1-e^{-z\log p}}$ in a convergent series of powers of $e^{-z\log p}$,

i.e. $$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p\in P} \log p \sum_{k=1}^{\infty} \left(e^{-z\log p}\right)^k.$$

Since the convergence of the series is absolute:

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{k=1}^{\infty} \sum_{p\in P} \log(p)\, e^{-z\log p^k}$$

$$= \sum_{n=2}^{\infty} \Lambda(n)\, e^{-z\log n}.$$

<u>Holomorphicity</u>: Let us prove that, if Φ is holomorphic in a neighborhood of $P\in\mathbb{R}$, if Φ is not zero, and if $\Phi(P)=0$, then $\mathrm{Re}\,\dfrac{\Phi'(z)}{\Phi(z)}>0$ for $z\in\mathbb{R}$ near P and to the right of P.

Let's start by assuming

$$\Phi(z) = \alpha (z-\rho)^k + \dots, \quad k \geq 1$$

such that $\Phi'(z) = k\alpha (z-\rho)^{k-1} + \dots$

and $\mathrm{Re}\,\dfrac{\Phi(z)}{\Phi'(z)} = \mathrm{Re}\left(k(z-\rho)^{-1} + \dots\right) > 0$

Zeros at the boundary of the critical strip

Here we will demonstrate that $\zeta(z)$ possesses no zeros at the boundary of the critical strip. By the functional eq., it is enough to show that there are no zeros on $\{z : \mathrm{Re}\, z = 1\}$.

Suppose that $\zeta(1+it_0) = 0$ for some $t_0 \in \mathbb{R}$, $t_0 \neq 0$. Let us define

$$\Phi(z) = \zeta^3(z)\, \zeta^4(z+it_0)\, \zeta(z+2it_0)$$

We can observe that ϕ has a zero at $z=1$ since ζ^3 has a 3rd order pole at $z=1$ while ζ^4 has a zero of order at least four at $1+it_0$.

But
$$\mathrm{Re}\ \frac{\phi(x)}{\phi'(x)} > 0 \quad \text{for } 1 < z < 1+\varepsilon_0$$
for some $\varepsilon_0 > 0$.

On the other hand, we have

$$\frac{\phi'(x)}{\phi(x)} = 3\frac{\zeta'(x)}{\zeta(x)} + 4\frac{\zeta'(x+it_0)}{\zeta(x+it_0)}$$

$$+ \frac{\zeta'(x+2it_0)}{\zeta(x+2it_0)}$$

$$= \sum_{n\geq 2} \Lambda(n) \Big\{ -3e^{-x\log n} - 4e^{-(x+it_0)\log n}$$

$$- e^{-(x+2it_0)\log n} \Big\{$$

As a result, we have

$$\operatorname{Re}\frac{\phi'(x)}{\phi(x)} = \sum_{n\geq 2} \Lambda(n)\, e^{-x\log n}\{-3 - 4\cos(t_0\log n) - \cos(2t_0\log n)\}$$

$$= \sum_{n\geq 2} \Lambda(n)\, e^{-x\log n}\{-3 - 4\cos(t_0\log n) - (2\cos^2(t_0\log n) - 1)\}$$

$$= -2\sum_{n\geq 2} \Lambda(n)\, e^{-x\log n}(\cos(t_0\log n) + 1)^2 \leq 0,$$

contradicting our initial assumption that $\operatorname{Re}\frac{\phi'(x)}{\phi(x)} > 0$.