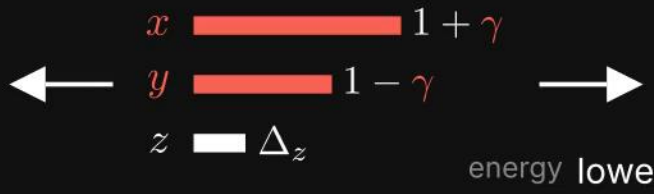
Part 2 left you here



χ_{zz}?



Positive coupling: neighbours prefer to point opposite.

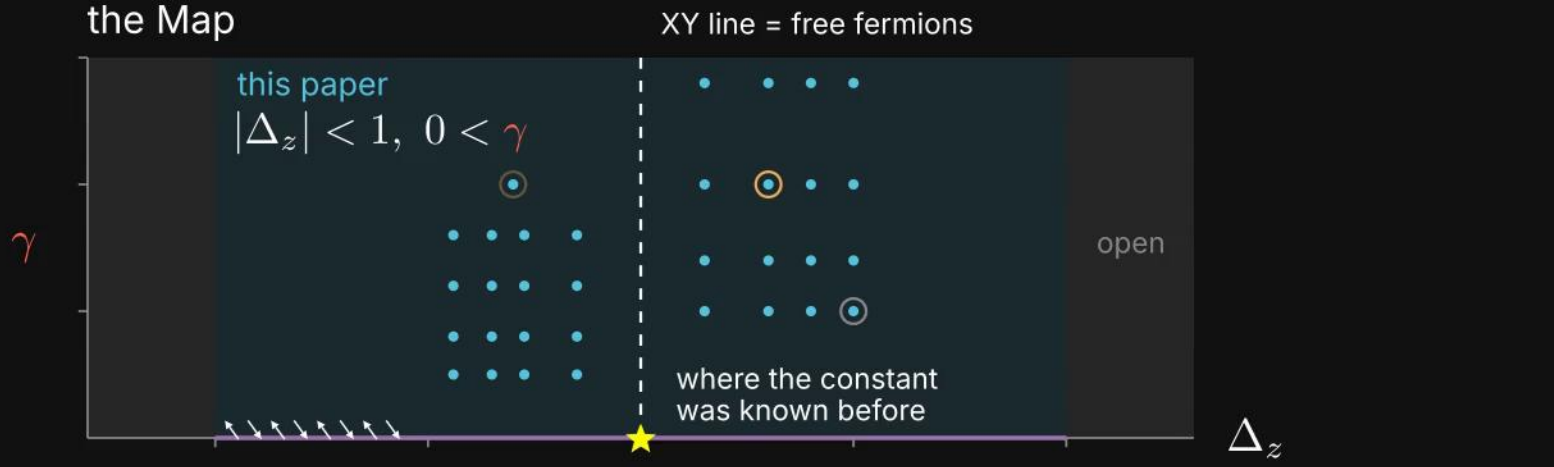
length of a rod = how much the pair cares about that axis

$$H = \sum_j \left[(1+\gamma)\,\sigma_j^x \sigma_{j+1}^x + (1-\gamma)\,\sigma_j^y \sigma_{j+1}^y + \Delta_z\,\sigma_j^z \sigma_{j+1}^z \right]$$



? One knob, gamma. Which of its consequences does the meter care about?

the Map

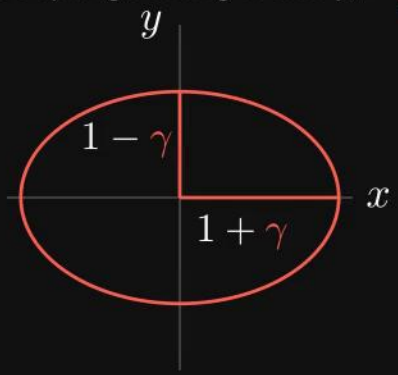


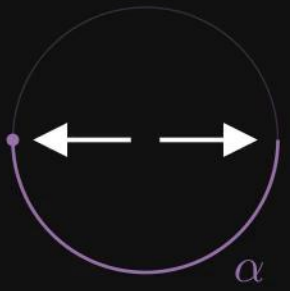
dots: the paper's runs
gold ring: home
grey ring: the hardest point

BLACK BOX · `Jordan-Wigner (XY line = free fermions)`

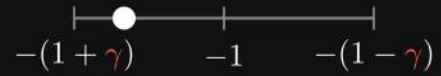
- On the line Δz = 0 a change of variables turns the spins into fermions that do not interact; stated, not shown.
- Two exact results live on that line and feed 3.7 and 4.5: the kink's inverse size and the number under Part 2's question mark.
- Off the line the fermions interact; nothing in these talks relies on the fermion picture.

couplings along x and y, γ





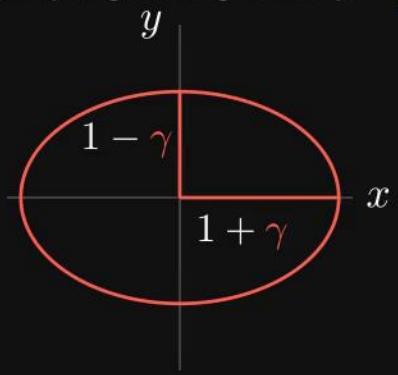
$\langle H \rangle$ per bond, product state

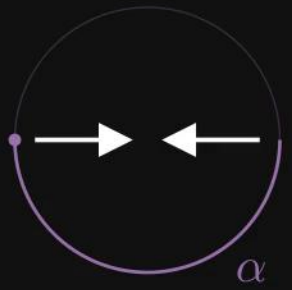


the energy knows where x is

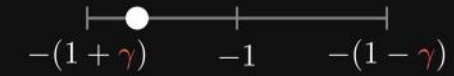
a product state, not the ground state

couplings along x and y, γ





$\langle H \rangle$ per bond, product state



the energy knows where x is

a product state, not the ground state

$$\sigma^x\sigma^x - \sigma^y\sigma^y = 2(\sigma^+\sigma^+ + \sigma^-\sigma^-)$$

$$[H, Q] \neq 0 \iff \gamma \neq 0$$

the red-corner factory

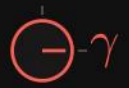


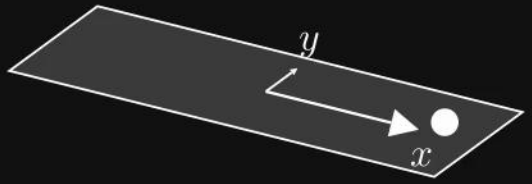
Q = Part 2's Q_A with the window removed

γ moves the chain's charge in steps of two.

explicit

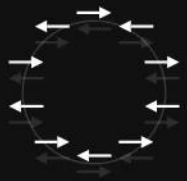
the tilted table





γ tilts the table: the rule itself prefers x

$$[H, Q] \neq 0$$



a finite ring returns the cat:
both falls at once;
we pin one (Caveats)

spontaneous

the falling pencil



the rule is even-handed between $+x$ and $-x$



same energy same energy

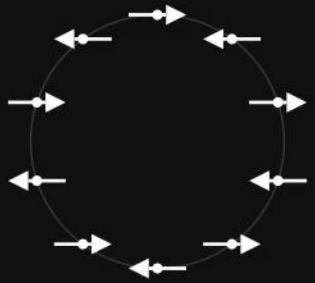
$$F = \prod_j \sigma_j^z$$

$$[H, F] = 0, \;\; F|\mathrm{GS}\rangle \neq |\mathrm{GS}\rangle$$

Tilted table (explicit): γ tilts the rule itself; $[H, Q] \neq 0$.
Falling pencil (spontaneous): the rule is even-handed,
the state picks a side; $[H, F] = 0, \;\; F|\mathrm{GS}\rangle \neq |\mathrm{GS}\rangle$.
The $\frac{1}{2} \log \ell$ growth needs the tilted table; the numerics sit in one fall of the pencil.

$E_2 - E_0 > 0, \quad E_1 - E_0 \to 0$ (two vacua)

a small ring of spins



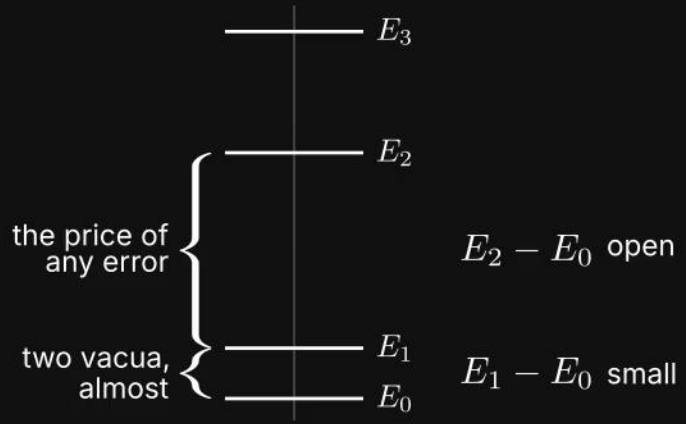
γ



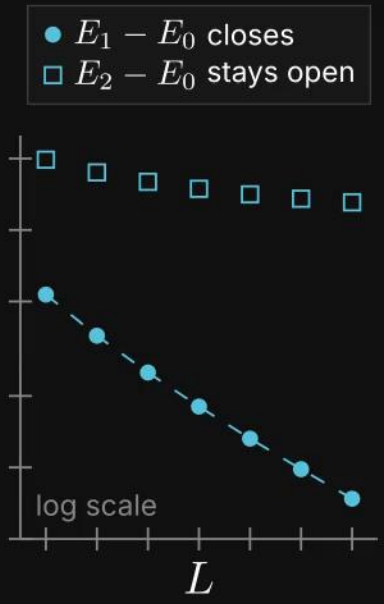
γ turns on

Δ_z fixed

the energy ladder



energies in units of the couplings;
never compared with M

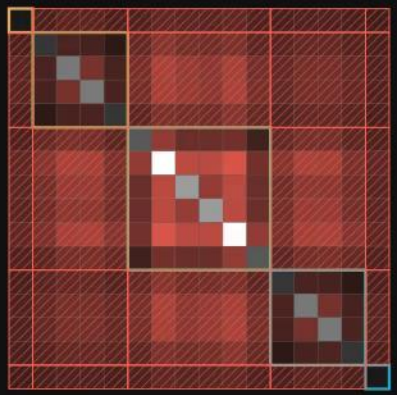


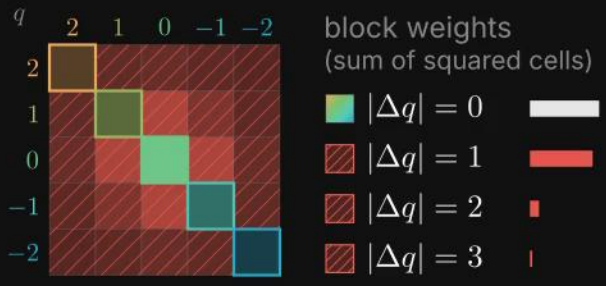


$$\langle \sigma_j^x \rangle = (-1)^j m_x, \quad \langle \sigma_j^z \rangle = 0 \quad 0 < m_x < 1$$

the order parameter

along z every spin is a fair coin; the stripes live in the red corners





all blocks shaded; $|\Delta q| = 1$ dominates

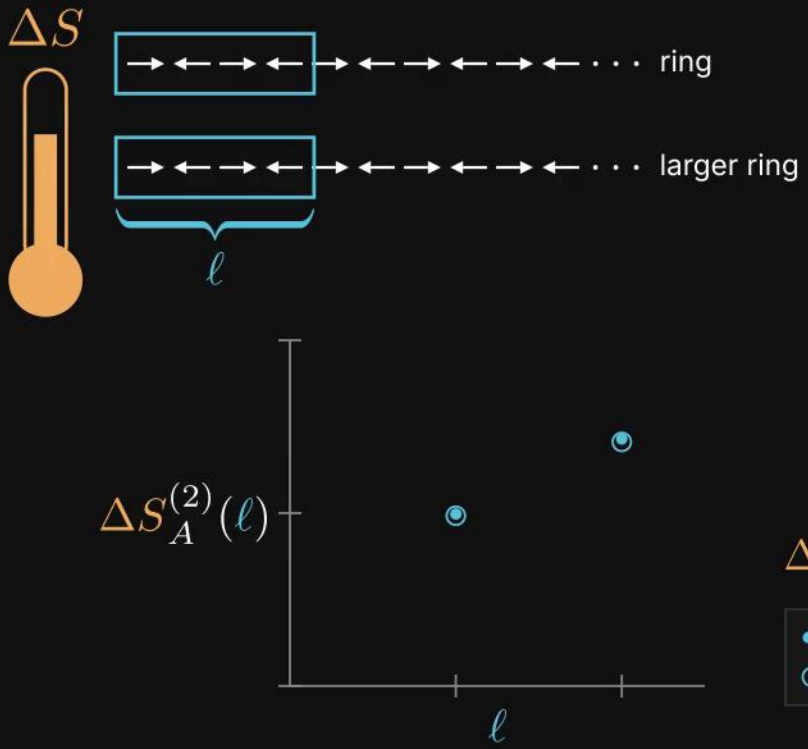
two apart (even): $F = \prod_j \sigma_j^z$

allowed by the symmetry γ leaves behind: the charge moves in steps of two, so even the symmetric, unfallen ring state has them already

one apart (odd):

a needle along x is a corner between up and down, one spin at a time: these need the fall of the pencil

Caveats: pinned = the combination of the two lowest ring states with the largest overlap with the Neel-x product state



? PAUSE AND PONDER

Turn the knob off (γ = 0, same Δz). What does the meter read on the symmetric ground state?

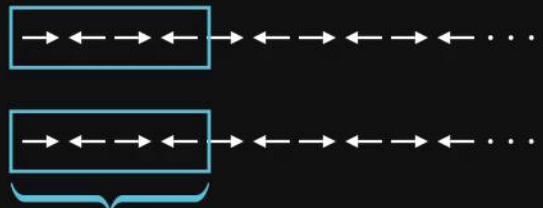
Vote: Still a reading like before

- Exactly zero, for every window
- It depends on the ring size

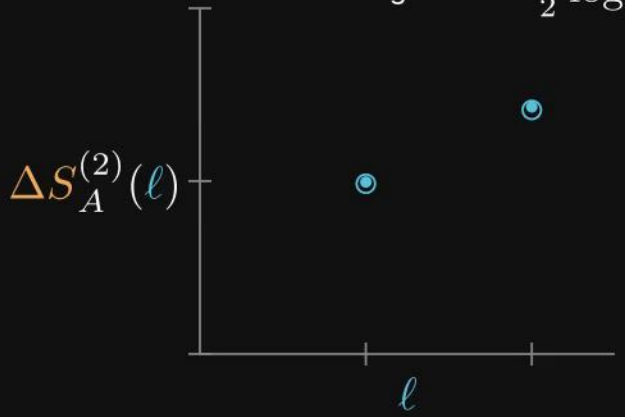
$\Delta S_A^{(2)}(\ell)$ from ED ρ_A

- ring
- larger ring





the meter cares about the tilted table;
whether it grows like $\frac{1}{2}\log \ell$ is Part 4's test



$\Delta S_A^{(2)}(\ell)$ from ED ρ_A

- ring
- larger ring

? PAUSE AND PONDER

Turn the knob off ($\gamma = 0$, same Δz). What does the meter read on the symmetric ground state?

Vote: Still a reading like before
| Exactly zero, for every window
| It depends on the ring size

symmetric ground state

$\gamma = 0$, same Δ_z

window	ΔS
short	0
longer	0
every one	0

flat table: one definite charge, no corners between sectors

exact: a definite charge has no corners between sectors



does spin r remember spin 0's z?

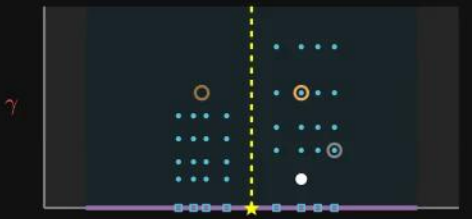


the knob



M
set by the couplings

the Map

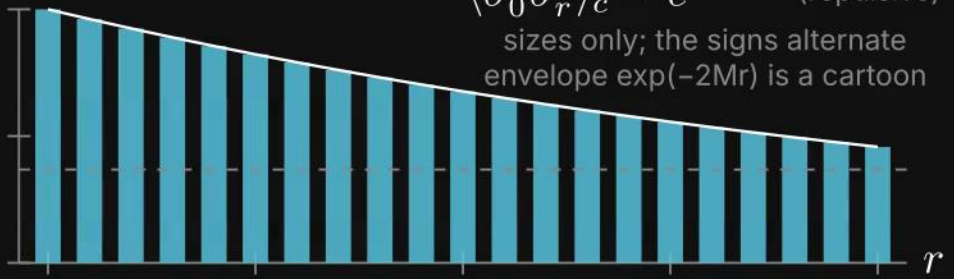


at $\gamma = 0$ the memory is infinite:
the gapless edge of the Map

$|\langle \sigma_0^z \sigma_r^z \rangle_c|$

$\langle \sigma_0^z \sigma_r^z \rangle_c \sim e^{-2Mr}$ (repulsive)

sizes only; the signs alternate
envelope exp(−2Mr) is a cartoon



pair-correlation length $1/(2M)$

kink size $1/M$

M is an inverse length:
$1/M$ counts sites

does spin r remember spin 0's z?

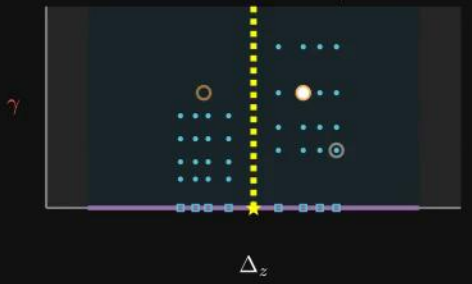


the knob



M
set by the couplings

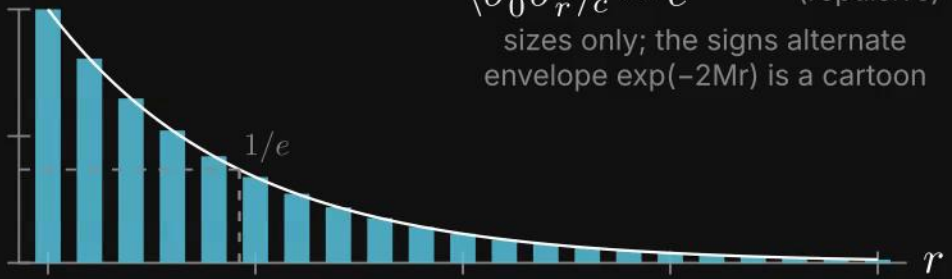
XY line $M = \operatorname{atanh}\gamma$



$|\langle\sigma_0^z\sigma_r^z\rangle_c|$

$\langle\sigma_0^z\sigma_r^z\rangle_c \sim e^{-2Mr}$ (repulsive)

sizes only; the signs alternate
envelope exp(−2Mr) is a cartoon



pair-correlation length $1/(2M)$

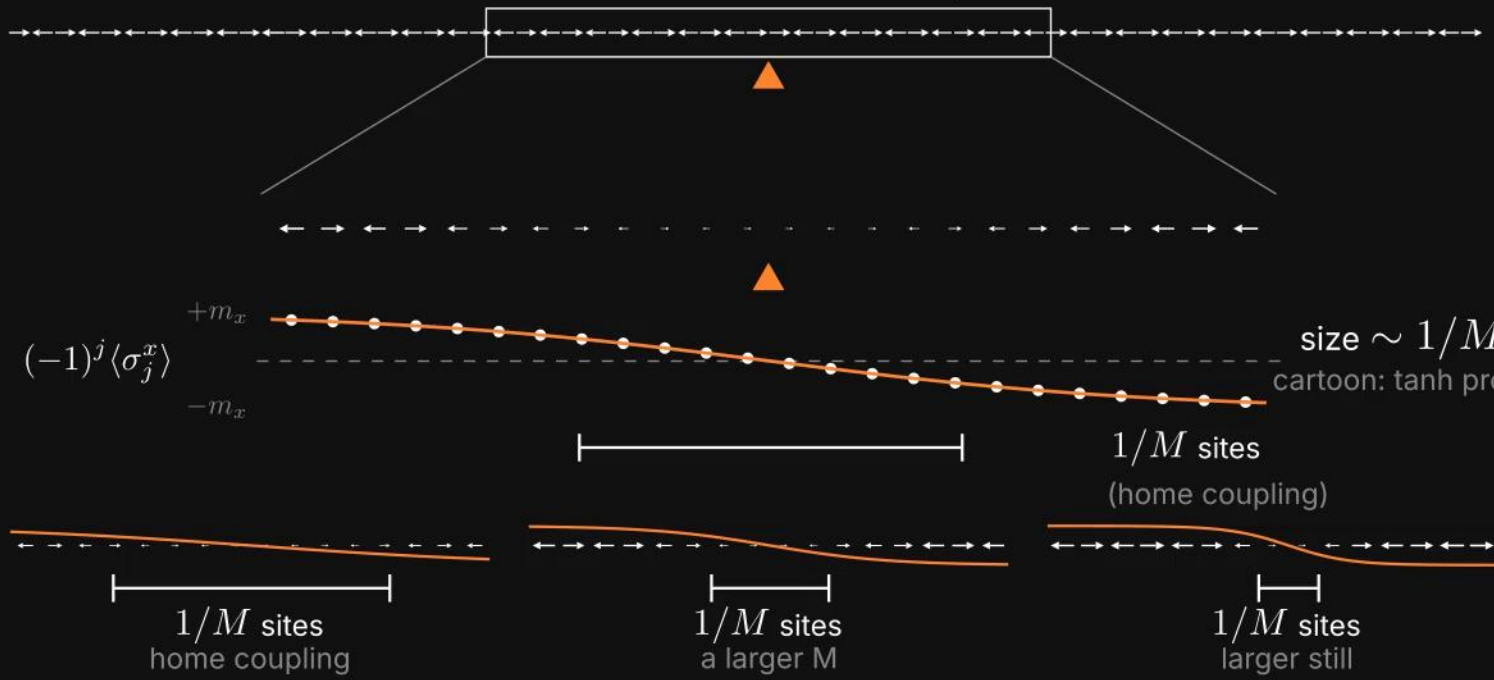
kink size $1/M$

M is an inverse length:
$1/M$ counts sites

BLACK BOX · Baxter/JKM: M known exactly

- Baxter (1972) and Johnson, Krinsky and McCoy (1973) solved this chain; M follows from their formulas at any Δz, γ of the strip $|\Delta z| < 1$, $\gamma > 0$ (elliptic functions, Caveats); card again in 4.8.
- On the XY line it reduces to M = atanh γ.
- Every M in these talks is this exact value, never a fit.

a wall between the two falls of the pencil: a kink



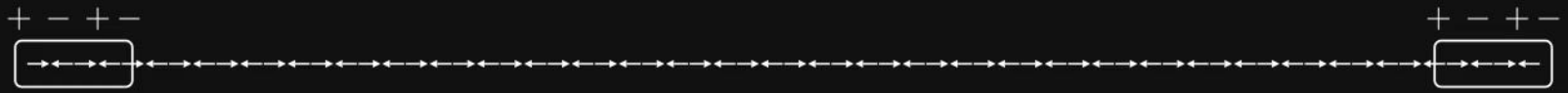
M is one over a length; an energy needs a speed

$$E_{\text{kink}} = v\,M$$

energy cost $= v\,M$; v set by the couplings

walls: 0 ▲ kink ▲ antikink

cartoon: a pair in flight
at the home coupling



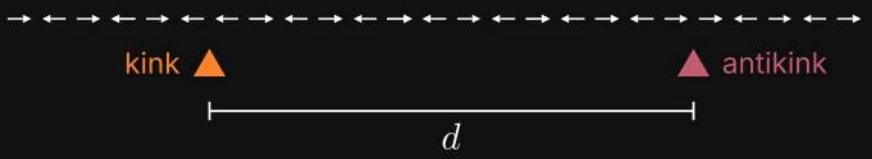
cartoon

$$\sigma_j^z \left| \rightarrow\leftarrow\rightarrow\leftarrow \cdots \right\rangle \rightarrow \text{kink} + \text{antikink}$$

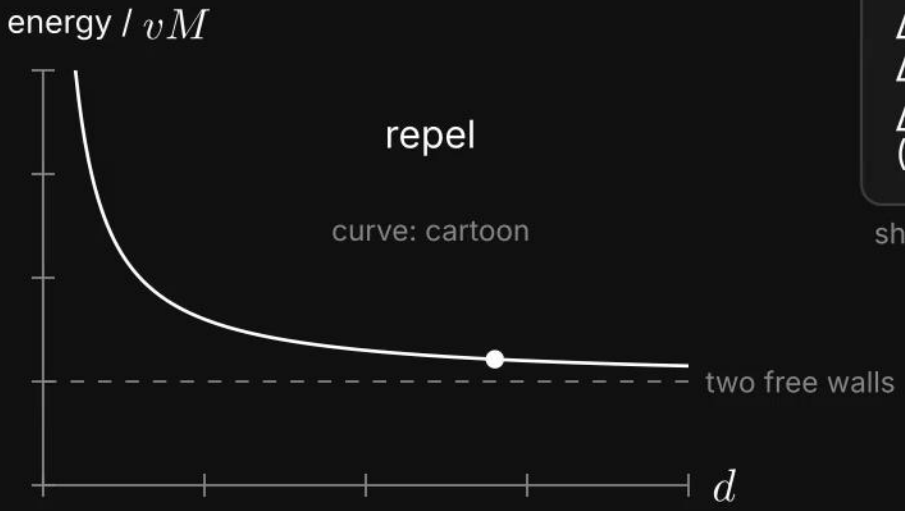
A single σ^z makes walls in pairs.

far left and far right never change; the number of walls between them is always even (Caveats)

at home, where walls repel, a pair costs $2\,vM$



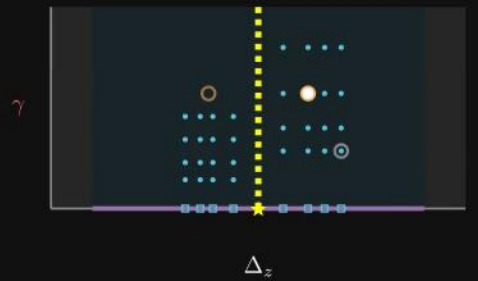




Δz < 0: attract.
Δz > 0: repel.
Δz = 0: ignore each other (free fermions).

shapes are cartoons; the sign is the physics

the Map



at ξ = 1 the pair is no longer bound





cartoon: a free kink and antikink drift apart



$$m_{B_1} = 2M \sin \frac{\pi\xi}{2} \quad (\xi < 1)$$

two free walls and above

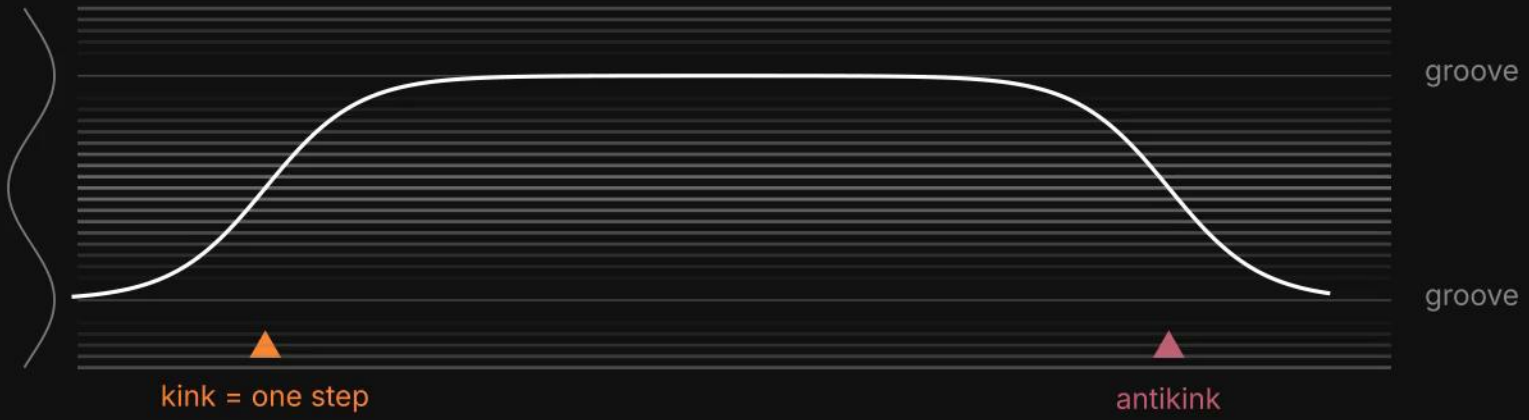
the bound pair $2\sin(\pi\xi/2)$

Physicists relabel Δz as ξ:
ξ = 1 free, ξ < 1 attractive, ξ > 1 repulsive.
Home: Δz > 0 sits at ξ > 1.

scaling-limit law ($\gamma \to 0$); on the lattice the ratio $m_{B_1}/2M$ sits a little below $\sin(\pi\xi/2)$
only the first bound state exists in the paper's range of ξ;
more appear only below ξ = 1/2



the recipe Δz → ξ is on the Caveats



cartoon: the needles' angle as a string; each groove is one fall of the pencil

$$m_{B_1} = 2M \sin \frac{\pi\xi}{2} \quad (\xi < 1)$$

BLACK BOX · `sine-Gordon`

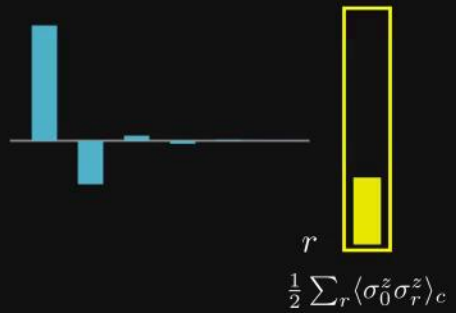
- As γ → 0 the stripes' angle becomes a smooth field in a cosine washboard; kinks are its steps between grooves; the recipe Δz → ξ is on the Caveats.
- The bound-pair law holds in that limit; on the lattice the ratio m_B1/2M sits a little below sin(πξ/2) and drifts down as γ grows.
- Part 4 borrows from it the bound-pair law above and what Lukyanov found for how a kink pair and the bound pair couple to the charge (4.7).

χzz is called a susceptibility.
For a conserved charge at temperature T, the answer to a small field is the width per site divided by T.
Here the charge is not conserved: χzz is strictly the width per site, at T = 0.

$$\chi_{zz} = 2 \lim_{\ell \to \infty} \mathrm{Var}(Q_A)/\ell = \frac{1}{2} \sum_r \langle \sigma_0^z \sigma_r^z \rangle_c$$

the variance is the bars added up, times ℓ/2, up to the edges

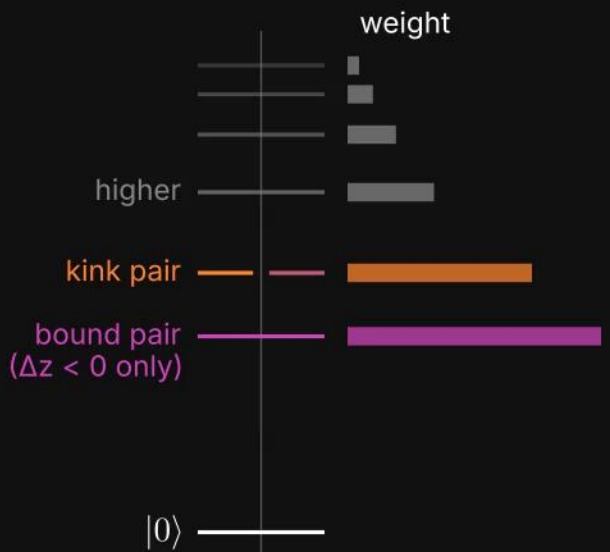
$\langle \sigma_0^z \sigma_r^z \rangle_c$ small ring



infinite chain
(iDMRG, next scene)

a small ring cannot hold the chain's memory: a toy

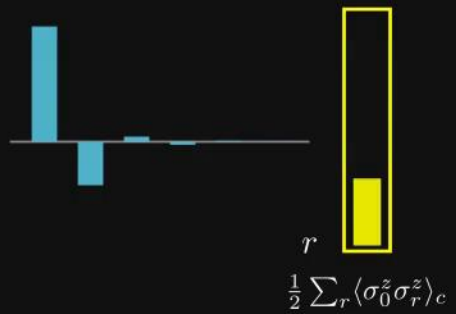
$$\mathrm{Var}(Q_A) = \sum_{k \neq 0} |\langle k|Q_A|0\rangle|^2$$



cartoon: rungs and weights not computed;
the bound pair exists only at Δz < 0 (3.10b)

a sum over excited states;
Part 4 shows why the lightest weigh most

$\langle \sigma_0^z \sigma_r^z \rangle_c$ small ring



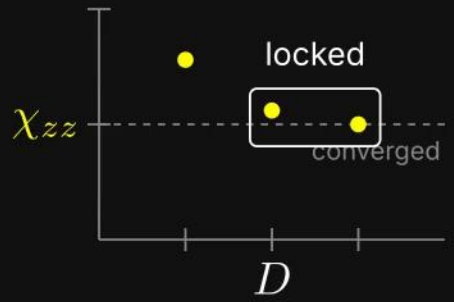
infinite chain
(iDMRG, next scene)

a small ring cannot hold
the chain's memory: a toy

a list of 2^L numbers: 2^L $2^{L+1} = 2 \cdot 2^L$ $2^{2L} = (2^L)^2$



JPEG for wavefunctions.



BLACK BOX · `iDMRG (everything inside)`

- The train of tables is a matrix-product state (formula on the Caveats); "grow" and "compress" are its two moves, run with TeNPy in the paper.
- Keeping the largest D eigenvalue bars at every cut is why it works for gapped chains (area law, 1.9).
- Convergence is judged by raising D until the digits stop moving; χzz is read from correlator sums in the converged state.

TAKE-HOME

$$\chi_{zz} = 2 \lim \mathrm{Var}(Q_A)/\ell = \tfrac{1}{2} \sum_r \langle \sigma_0^z \sigma_r^z \rangle_c$$

- γ tilts the table (explicit, [H, Q] ≠ 0) and manufactures the red corners; the stripes are one fall of the pencil (spontaneous); the meter cares about the tilt.
- A gap buys a short memory: correlations fade over 1/(2M) sites, errors are kinks of size 1/M costing vM, made in pairs; M is exact (Baxter) and is an inverse length.
- χzz is the width of the charge bell per site: twice the slope of the variance against ℓ, or half the sum of the z correlations over every distance; computed at home by iDMRG.

CAVEATS

- ☐ Jordan-Wigner: “XY line = free fermions” taken on faith (3.1b).
- ☐ The finite ring returns the cat (both falls at once); the pinned state is the combination of the two lowest ring states with the largest overlap with the Neel-x product state (scene 3.3, scene 3.5).
- ☐ Ring energies are in units of the couplings and are never compared with M; the ladder energies do not depend on the pinning (3.4).
- ☐ Gap ⇒ exponential decay (Hastings-Koma 2006) and Baxter/JKM's exact M (elliptic functions) taken on faith; on the attractive side the fading length is one over the bound pair's mass, not 1/(2M) (3.7).
- ☐ A kink's energy is vM; the speed v depends on the coupling: on the XY line it comes from the free-fermion dispersion, elsewhere from the Bethe ansatz (3.8).

CAVEATS

- ☐ “A single σ^z makes walls in pairs”: the signed count of walls is conserved because the stripes at infinity cannot change (topological charge) (3.9).
- ☐ Recipe (formula below, left) gives ξ from Δz; home sits at $\xi > 1$; the bound-pair law is the $\gamma \to 0$ limit; only B_1 exists in the paper's range of ξ (3.10b).
- ☐ χzz at home is an iDMRG value; the ring accumulator is a toy and never joins it; the 3.11 word said once, as width per site at $T = 0$; for this non-conserved charge the answer to a small field is a different number at every temperature (3.11).
- ☐ “Variance per site” is the talk's name for χzz, a deliberate simplification: χzz is twice the long-window limit of the charge variance divided by ℓ, so the variance per site is strictly $\chi zz/2$ (3.11).
- ☐ Matrix-product state (formula below, right): one D × D matrix per site, the amplitude is their trace; convergence judged in D at the hardest point of the Map (3.12).

$$\theta_0 = \arccos \Delta_z, \quad \beta^2 = 1 - \theta_0/\pi, \quad \xi = \beta^2/(1-\beta^2) \qquad |\Psi\rangle = \sum_s \mathrm{Tr}[A^{s_1} A^{s_2} \cdots]\, |s_1 s_2 \cdots\rangle$$

$$\Delta S_A^{(2)} = \tfrac{1}{2}\log(4\pi V), \quad V \stackrel{?}{=} \tfrac{1}{2}\chi_{zz}\ell$$

χ_{zz} (measured by iDMRG)

M (inverse size, exact)

The chain has a measurable variance per site χzz and an exactly known inverse size M. Is V = χzz ℓ/2 for a mixed piece? Can the lightest particles bound χzz/M from below with nothing fitted?