ingredients 1 2 3
tests 1 2 3



$\frac{1}{2}\log(4\pi V)$

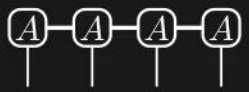


$\frac{\log n}{2(n-1)}$



χ_{zz}, variance per site

$\Delta S = ?$ $V = \boxed{?}$



the computer



the lightest particles

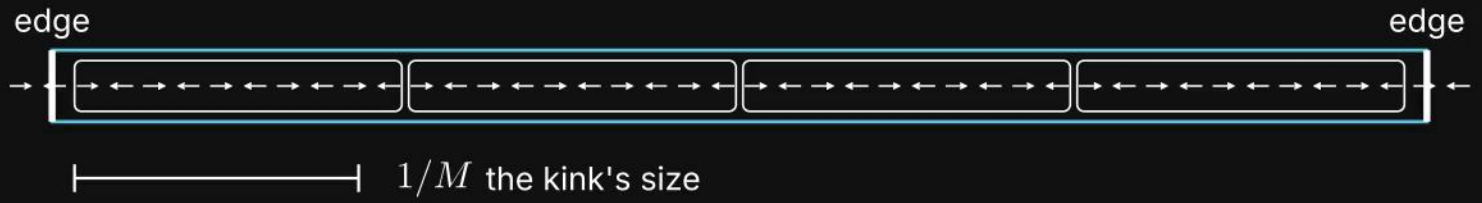


M, exact inverse length

? PAUSE AND PONDER

For a mixed window, is the Clock's width V equal to the variance of the charge Bell?

Vote: Yes, always | Only when the window is pure | Never for a mixed window



chunks longer than 1/M are independent: logs add;
both widths grow by the same amount per chunk;
only the edges differ

BLACK BOX · `transfer matrix ("logs add")` $\log Z_2(\alpha), \quad \log\langle e^{i\alpha Q_A}\rangle$

- For the chain's ground state as the computer stores it (3.12), both of these logs grow linearly with ℓ, and their curvatures at α = 0 give the same width per site: chunks longer than 1/M are independent, and the logs of independent pieces add.
- Both rates come from one object, the leading eigenvalue of a phase-dressed transfer matrix; what is left between V and the Bell's variance is an edge term that does not grow with ℓ.
- Proved in the paper for free fermions and for the computer's class of states; taken on faith here.

ingredients 1 2 3
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$$\Delta S_A^{(n)}(\ell) = \tfrac{1}{2}\log(\pi \ell \chi_{zz}) + \frac{\log n}{2(n-1)} + \mathcal{O}\left(\frac{1}{M\ell}\right)$$

Blue and yellow: half the log of pi, times the window, times the variance per site.
Gold: a pure number from the Dial; one half at n to 1, half the log of two at n = 2.
White: the edge term, a kink's length over the window's length.
Nothing fitted; on faith: “logs add” (4.2) and the walk from integer n to n = 1 (Black-box card).

BLACK BOX · `the n → 1 continuation`

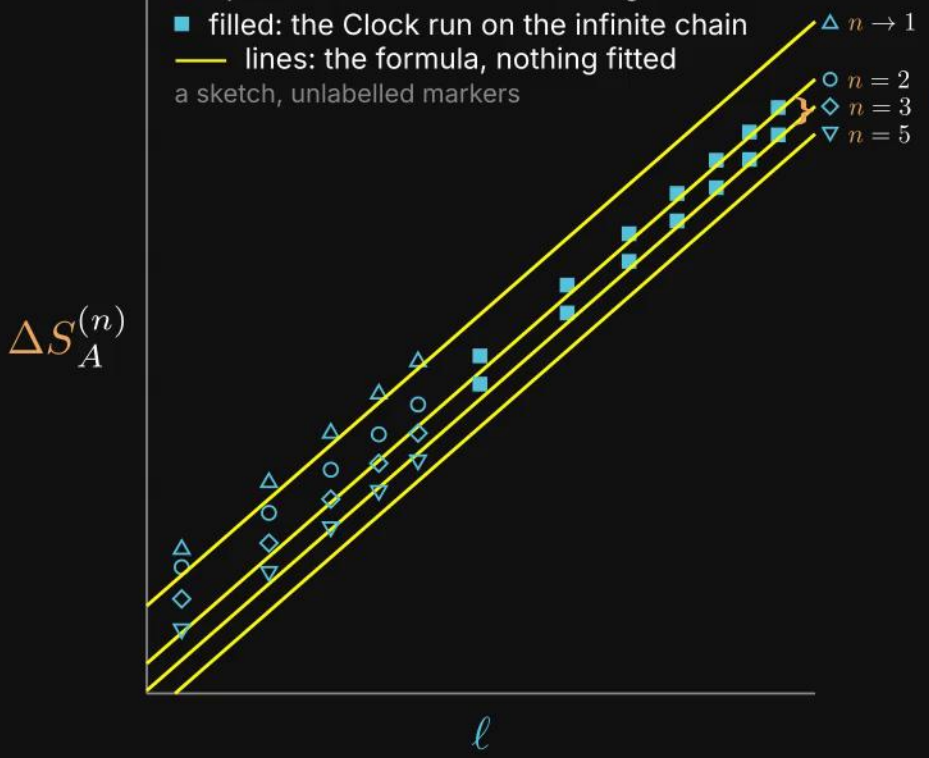
- Card again from 2.11: the gold offset log n / (2(n – 1)) is derived for every integer n ≥ 2, a Gaussian integral over n replicas placed on a circle.
- The von Neumann value is that formula continued to n → 1, where it reads 1/2; it is checked against the n → 1 points of Fig. 2(a) in 4.4, not derived.
- Behind it: all replica widths are equal, V_d = V; proved for the computer's class of states, with an edge term that stays finite as the window grows.

Test 1 passed: slope one half, offsets from the Dial, nothing fitted.

ingredients 1 2 3
tests 1 2 3

- open: window tables with charge drawers
- filled: the Clock run on the infinite chain
- lines: the formula, nothing fitted

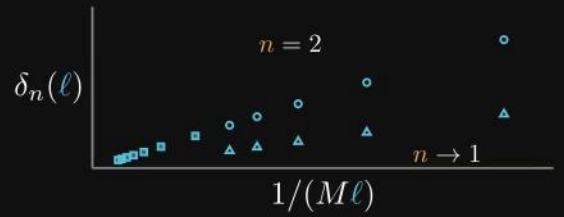
a sketch, unlabelled markers

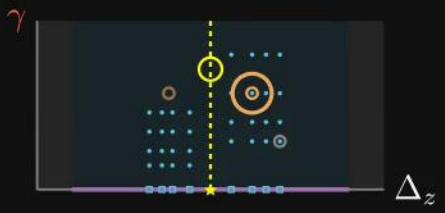


$$\text{gap} = \tfrac{1}{2}\log 2 - \tfrac{1}{4}\log 3$$

measured on the same state:
the same gap

$$\delta_n(\ell) \equiv \Delta S_A^{(n)}(\ell) - \left[\tfrac{1}{2}\log(\pi \ell \chi_{zz}) + \frac{\log n}{2(n-1)}\right] = \mathcal{O}\left(\frac{1}{M\ell}\right)$$





χ_{zz}
from the correlations

the two agree

$\chi_{zz} = e^{2c_1}/(2\pi)$
from the asymmetry

$$c_1 = \tfrac{1}{2}\log(2\pi\chi_{zz}) \;\Rightarrow\; \chi_{zz} = \frac{e^{2c_1}}{2\pi}$$

$c_1 < 0$ because $\chi_{zz} < 1/(2\pi)$

the attractive repeat: the measured c_1 and the formula's ½ log(2π χzz) agree
the free line: exact χzz = γ/(1+γ); the asymmetry road lands on it

At finite γ the free line's ratio χzz/M = (γ/(1+γ))/atanh γ sits below 1.
The star value 1 is the γ → 0 limit; the road to it is scene 4.10.



$$\chi_{zz} = \frac{2}{L} \sum_{k \neq 0} \frac{|\langle k|\dot{Q}|0\rangle|^2}{E_k^2}$$

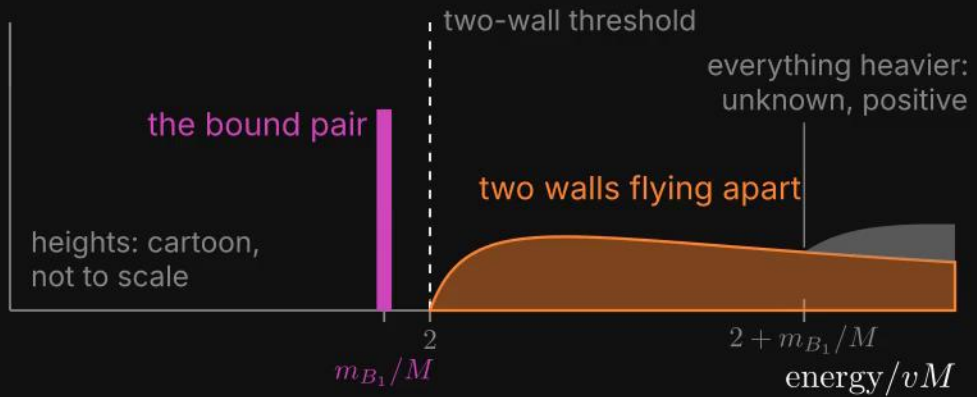
$$\dot{Q} = i[H, Q] \neq 0$$

the charge leaks

Every bar is positive: keep a few, and you hold a lower bound.

BLACK BOX · `the sum rule`

- Every excited state of the chain gets one bar; its height is the leak rate into that state, squared, divided by the state's energy squared; every bar is positive.
- The step from “variance” to “leak over energy” is one line: for an eigenstate, the matrix element of $\dot{Q}$ is iE_k times that of Q (Caveats).
- Which states are the lightest with a bar is decided by the dictionary of 3.10b; we take that list (two walls flying apart, and the pink rung) on faith; heavier states add more positive bars.



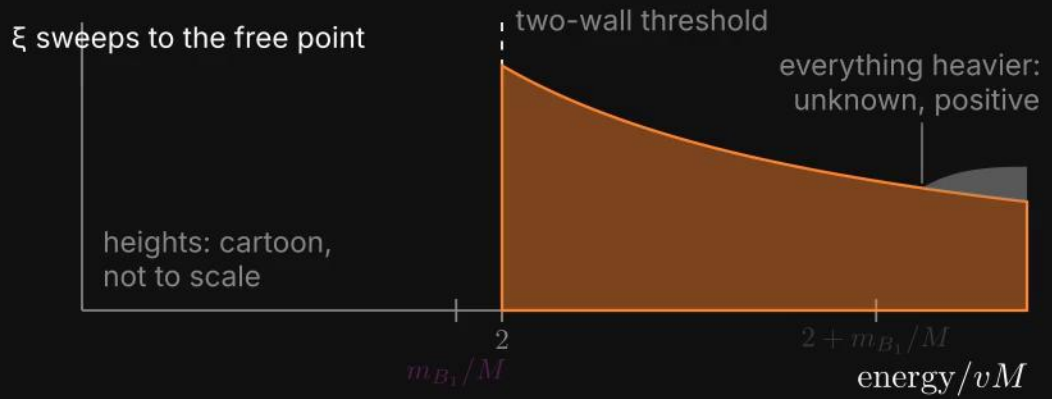


$$\mathfrak{c}(\beta^2) \equiv \chi_{zz}/M$$

named in 4.8b

$$\geq \mathfrak{c}_{2k} + \mathfrak{c}_{B_1}\Theta(1-\xi)$$

A lower bound, not a prediction.





$$\mathfrak{c}(\beta^2) \equiv \chi_{zz}/M$$

named in 4.8b

$$\geq \mathfrak{c}_{2k} + \mathfrak{c}_{B_1}\Theta(1-\xi)$$

A lower bound, not a prediction.

BLACK BOX · a function Lukyanov found

- The orange hump's shape is a function Lukyanov found in 1997 for the two-wall states of the continuum dictionary of 3.10b; its overall size is fixed by one more identity and one constant matched at the free point, not fitted.
- That identity is Hellmann-Feynman: the derivative of the ground-state energy with respect to the breaking coupling trades an unknown vacuum average for the inverse length M and the coupling, and reproduces a known exact energy.
- The pink bar's height is Lukyanov's one-breather value; its weight vanishes as ξ → 1 and its slope cancels the hump's threshold kink, a check the paper performs.

$$M = \operatorname{atanh} k_1'$$

k_1': an elliptic modulus (Caveats)

the home coupling

$\Delta_z = 0$

$\gamma = 0$

Baxter 1972 / JKM 1973

elliptic functions

$\Delta_z = 0$

M = exact

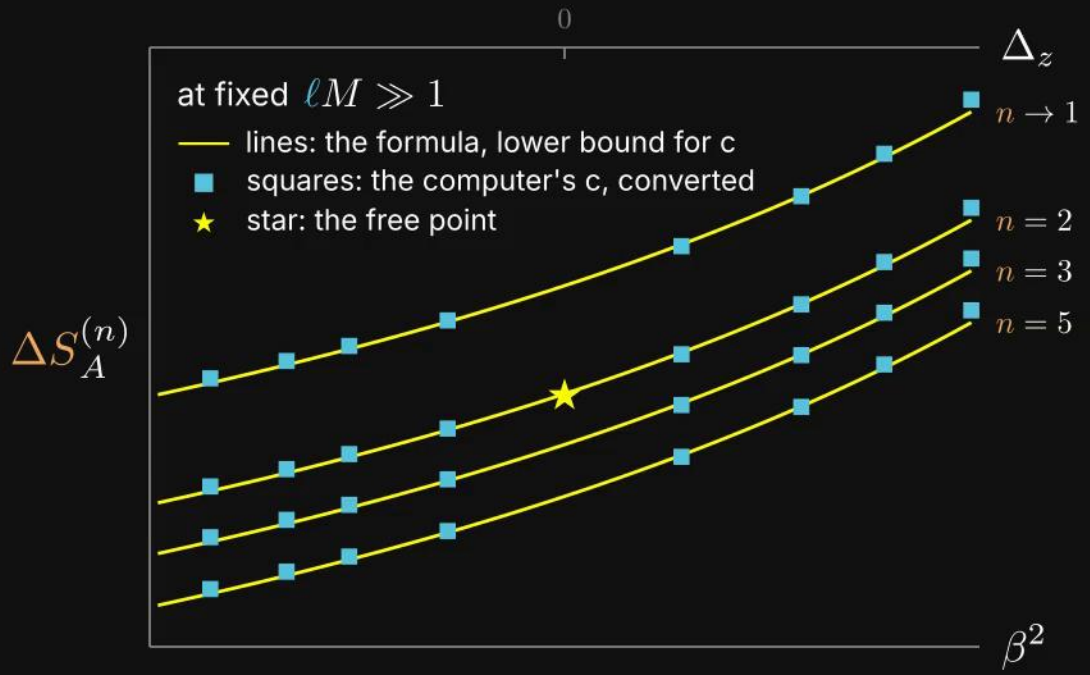


BLACK BOX · Baxter / JKM elliptic functions

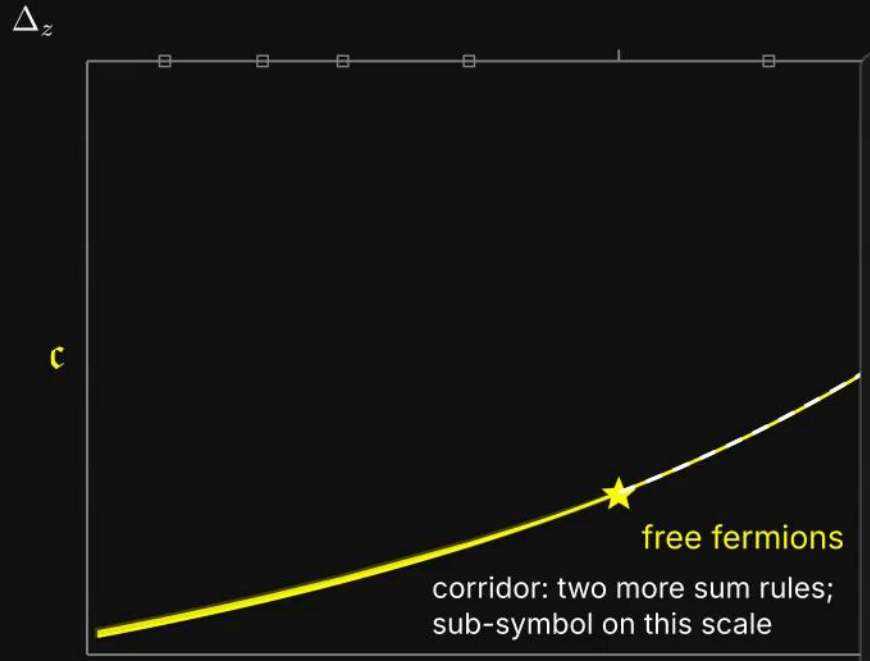
- Card again from 3.7: the machine is the exact solution of the eight-vertex model (Baxter 1972), read for the XYZ chain by Johnson, Krinsky and McCoy (1973); k_1' is an elliptic modulus built from Δz and γ.
- On the XY line every elliptic function reduces to one line and M = atanh γ; at small γ the output follows $M \propto \gamma^{(\pi/2\theta_0)}$ with a known prefactor (4.8b).
- We use only the output, exact to as many digits as we like; never a fitted mass.

$$\Delta S_A^{(n)} = \tfrac{1}{2}\log(\pi\ell) + \tfrac{1}{2}\log\left[\mathfrak{c}(\beta^2)\left(\frac{\gamma\, 4^{1-2\beta^2}}{\sin^2\theta_0}\right)^{\pi/2\theta_0}\right] + \frac{\log n}{2(n-1)}$$

ingredients 1 2 3
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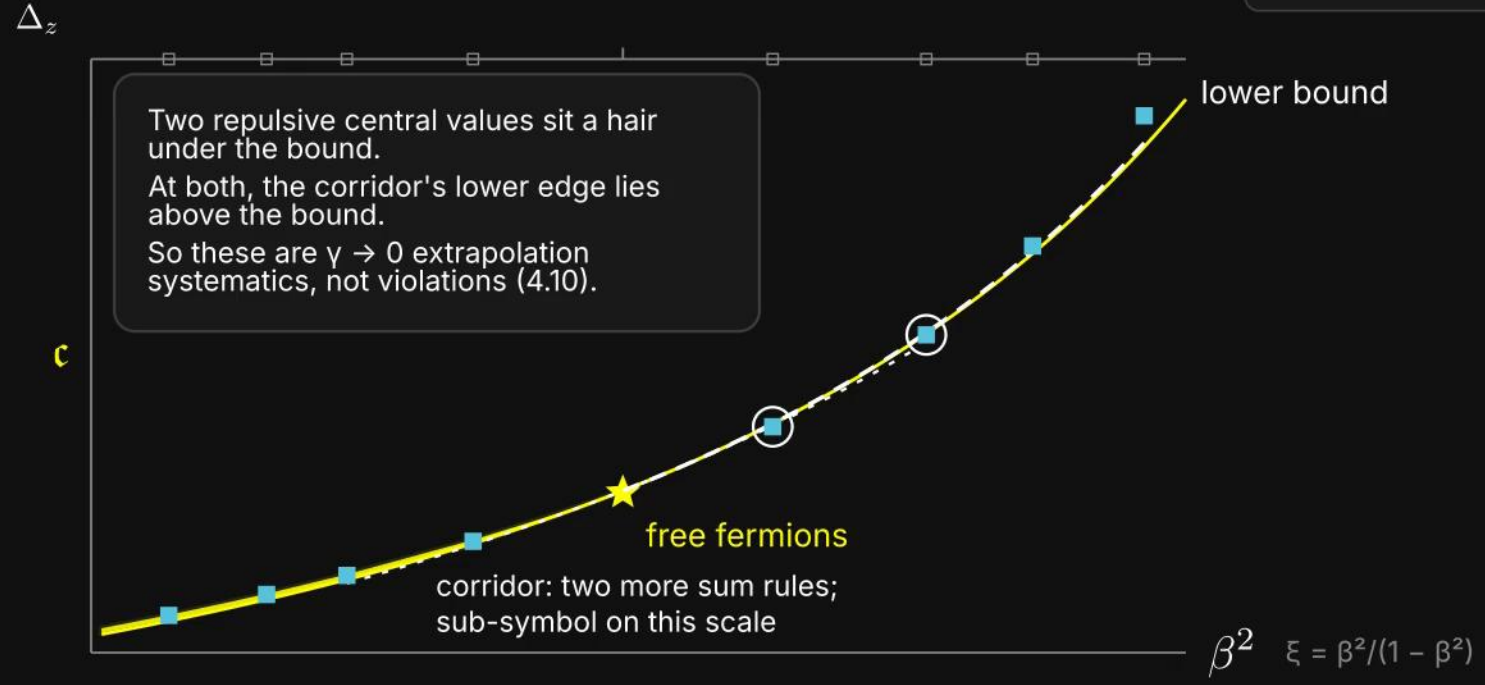


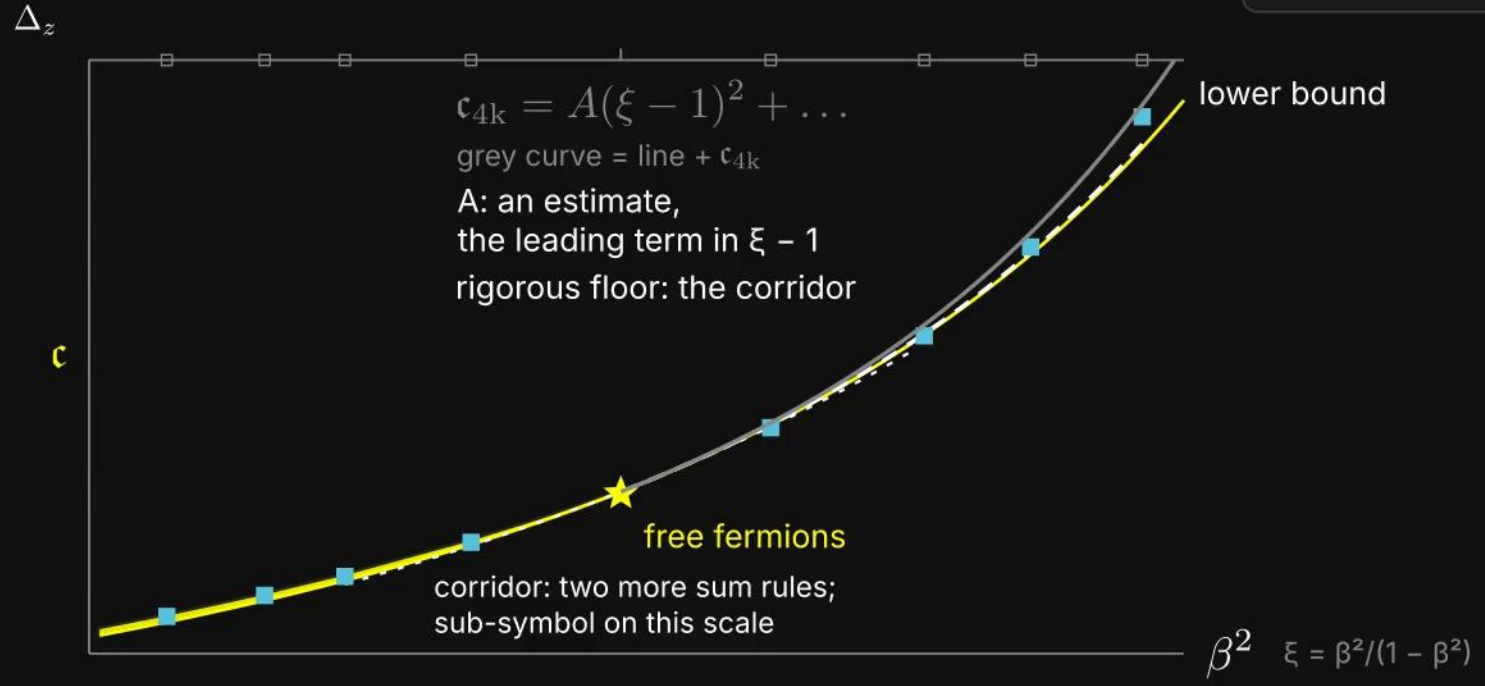
Eq. (final) holds for γ → 0 at fixed ℓM ≫ 1. Finite-γ corrections: a regular power series in γ, and a lattice-only fractional power $\gamma^{(4K-2)\pi/2\theta_0}$ with $K = \frac{1}{2\beta^2}$. Named here, used in the error bars (4.10), never shown.



BLACK BOX · the moment corridor

- Two more sum rules weigh the same bars with $1/E_n$ and with E_n; both totals are known exactly (a shift of the field for the first; an f-sum rule for the second).
- Subtract the bar and the hump from both, and what is left are two moments of the unknown tail; the lightest omitted threshold gives a lower edge on both sides, and Cauchy-Schwarz an upper edge, which survives only for $\xi < 1$, where the E_n total is finite.
- The exact forms and the tail moments live in the paper's appendix and on the Caveats; the band on screen is their output.


Δ_z
Two repulsive central values sit a hair under the bound.
At both, the corridor's lower edge lies above the bound.
So these are γ → 0 extrapolation systematics, not violations (4.10).
lower bound
c
free fermions
corridor: two more sum rules;
sub-symbol on this scale
β^2
ξ = β²/(1 − β²)

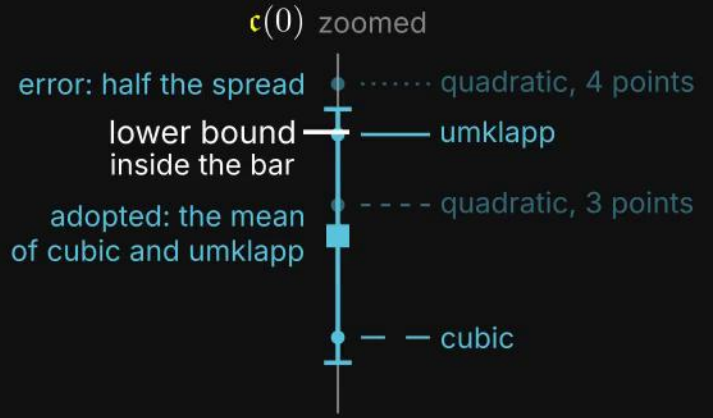


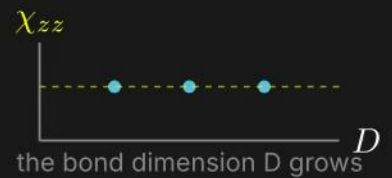
Test 3: a lower bound honoured within the uncertainties at eight couplings; the corridor holds at all four attractive ones.

$$\mathfrak{c}(\gamma) = \mathfrak{c}_0(1 + a\gamma + b\gamma^s)$$

BLACK BOX · the γ → 0 extrapolation forms

- Four roads: a quadratic through the first three points, a quadratic through all four, a cubic through all four, and the umklapp form (formula above) with the exponent s fixed by Δz (no free exponent).
- Adopted value: the mean of the cubic and umklapp roads; error bar: half the spread of the four.
- Why a fractional power at all: lattice operators absent from the continuum; the exponent is $(4K-2)\pi/2\theta_o$ with $K = 1/(2\beta^2)$ (Caveats).

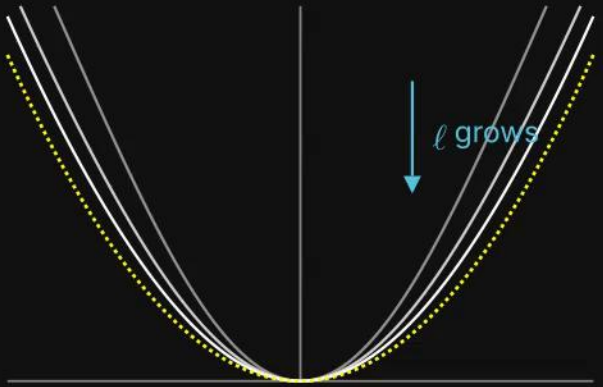




the computer's own error is far smaller than the road's

ingredients 1 2 3
tests 1 2 3

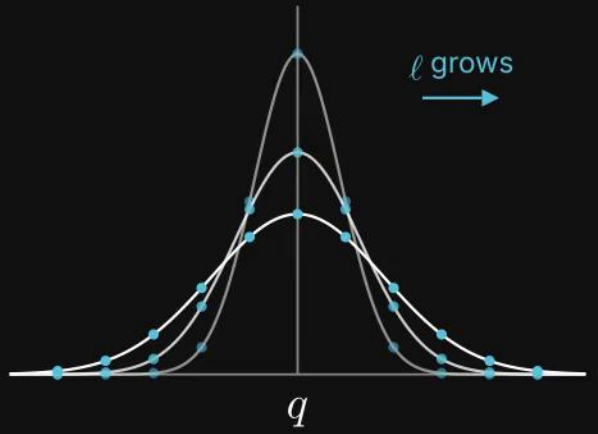
$-\log Z_2(\alpha)/\ell$



the coins



$P(Q_A)$

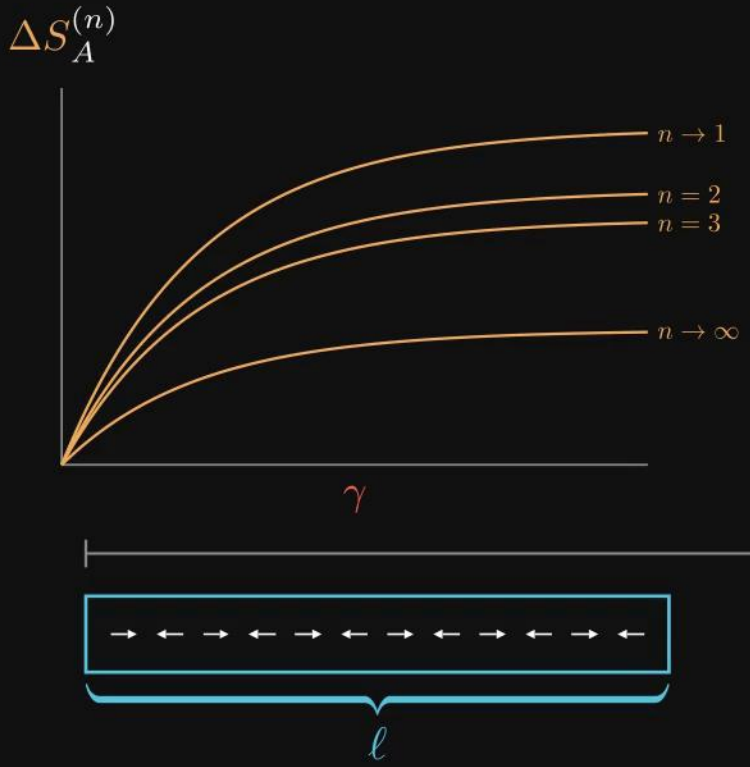


$$-\frac{\log Z_2(\alpha)}{\ell} \to \frac{\chi_{zz}}{2}\alpha^2$$

the Gaussian saddle of 2.10, measured

solid: bells of the measured variance, which tends to $\ell\chi_{zz}/2$

a sketch of the paper's Fig. 5



ingredients 1 2 3
tests 1 2 3

top to bottom: $n \to 1,\ 2,\ 3,\ \infty$

a sketch of the paper's Fig. 3

Fixed ℓ: the meter reads 0 at γ = 0 and rises with γ; the formula holds where 1/M < ℓ.
two limits, ℓ → ∞ and γ → 0, do not commute; this plot is the lattice echo of that

TAKE-HOME

$$\Delta S_A^{(n)} = \tfrac{1}{2}\log(\pi \ell \chi_{zz}) + \frac{\log n}{2(n-1)}$$

- The meter of Part 2 on the chain of Part 3 reads ½ log(πℓχzz) + log n / 2(n−1) plus an edge term of order 1/(Mℓ): half the log of the variance, nothing fitted, three tests passed.
- The variance per site is a budget spent over excited states; the two lightest particles, two walls and the pink rung, give a lower bound on χzz/M with no free parameter, and a corridor pins the rest on the attractive side.
- With Baxter's exact inverse length the formula closes in lattice variables (Eq. final): γ and Δz in, the asymmetry out, valid for γ → 0 and ℓM ≫ 1; the error bars are half the spread of four roads to γ → 0.

CAVEATS

- ☐ Eq. (V) was shown once; V = Var(Q in the window) × [1 + O(ℓ^{-1})] was taken on faith (“logs add”), proved in the paper for free fermions and injective matrix-product states only (4.2).
- ☐ Which of the two measured widths is V and which the window's variance is not established; only the common slope and the constant gap were used (4.2).
- ☐ Still on faith from Part 2: the 2023 definition and the quantum Mpemba effect, named only (2.1); $\Delta S^{(n)} \geq 0$, asserted, its one-line majorization proof on Part 2's Caveats (2.5).
- ☐ Still on faith from Part 3: Jordan-Wigner, “XY line = free fermions” (3.1b); gap ⇒ exponential decay, Hastings-Koma 2006 (3.7); the continuum dictionary of 3.10b and its bound-pair law; everything inside iDMRG, the computer behind every filled point and every square (3.12).
- ☐ The Dial's offsets come from the replica-loop Gaussian integral (2.11) and assume equal replica widths, V_d = V, proved for the computer's class of states; the n → 1 value is an analytic continuation checked against data (2.11, 4.3, 4.4).

CAVEATS

- ☐ The sum rule's one-line step from the variance to "leak rate squared over energy squared" was stated, not derived (4.6); which excited states carry a bar (two walls in the odd channel, the pink rung; not B_1B_1 or B_2) comes from the continuum dictionary of 3.10b; the hump's shape is "a function Lukyanov found" and was never written out; the Hellmann-Feynman step eliminates a vacuum average (4.6, 4.7).
- ☐ The line of Fig. 1 is a lower bound; the corridor needs two more sum rules whose closed forms were not shown; it is two-sided only for $\xi < 1$ (4.7, 4.9).
- ☐ The four-kink coefficient A is a Monte-Carlo estimate with sizeable truncation systematics; its rigorous floor comes from the corridor; the letter A here is the four-kink coefficient, not the window of 1.2 (4.9).
- ☐ M comes from elliptic functions (Baxter/JKM) we never opened; $M \propto \gamma^{(\pi/2\theta_o)}$ is its $\gamma \to 0$ limit and the finite-γ slopes differ; M is an inverse length, a kink's energy is vM (4.8).
- ☐ Eq. (final) holds for $\gamma \to 0$ and $\ell M \gg 1$; finite-γ corrections (a power series and a lattice-only fractional power) enter only through the extrapolation (4.8b, 4.10).

CAVEATS

- ☐ Error bars are half the spread of four fit forms with the mean of cubic and umklapp as centre; the two marginal repulsive points of Fig. 1 are extrapolation systematics and stay out of the finale (4.9, 4.10).
- ☐ On the free line at finite γ the ratio $\chi zz/M = (\gamma/(1+\gamma))/\operatorname{atanh}\gamma$ is below 1; the star is the $\gamma \to 0$ limit (4.5).
- ☐ Fig. 3 and Fig. 5 are shown as sketches of their structure, not as the published figures (4.11, 4.12).
- ☐ The free-line run behind the anchor is placed on the Map at a coupling inferred from the match, not recorded (4.5).
- ☐ Four helper symbols beyond the roadmap's four Part 4 symbols (V, owned by 2.9 and extended to mixed windows in 4.2; c, c_1, β^2): $\delta_n(\ell)$ (4.4), $\dot{Q}$ (4.6), $\theta_o = \arccos \Delta z$ (4.8, a Caveats symbol of 3.10b) and the four-kink A (4.9); each sits in its scene's roadmap equation and in the handout glossary.

Beyond the Neel-x phase: $\Delta z > 1$, where the breaking is staggered and the same marginality returns.

A field coupled to the broken charge.

After a quench: Eq. (final) is the equilibrium baseline for Mpemba-type behaviour.

The tail: resum the four-kink term; a convergent repulsive-side f-sum rule would close the corridor.

What does the meter read beyond the Neel-x phase, in a field, or after a quench? That is the research frontier.